Bessel potentials and optimal Hardy and Hardy-Rellich inequalities

Nassif Ghoussoub* and Amir Moradifam †
Department of Mathematics, University of British Columbia,
Vancouver BC Canada V6T 1Z2
nassif@math.ubc.ca
a.moradi@math.ubc.ca

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Abstract

We give necessary and sufficient conditions on a pair of positive radial functions V and W on a ball B of radius R in R^n, $n \geq 1$, so that the following inequalities hold for all $u \in C_0^\infty(B)$:

$$\int_B V(x)|\nabla u|^2 dx \geq \int_B W(x) u^2 dx,$$

and

$$\int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B \left(\frac{V(x)}{|x|^2} - \frac{V_r(|x|)}{|x|}\right)|\nabla u|^2 dx.$$

This characterization makes a very useful connection between Hardy-type inequalities and the oscillatory behaviour of certain ordinary differential equations, and helps in the identification of a large number of such couples (V, W) – that we call Bessel pairs – as well as the best constants in the corresponding inequalities. This allows us to improve, extend, and unify many results –old and new– about Hardy and Hardy-Rellich type inequalities, such as those obtained by Caffarelli-Kohn-Nirenberg [9], Brezis-Vázquez [8], Wang-Willem [27], Adimurthi-Chaudhuri-Ramaswamy [1], Filippas-Tertikas [13], Adimurthi-Grossi -Santra [2], Tertikas-Zographopoulos [24], and Blanchet-Bonforte-Dolbeault-Grillo-Vasquez [4].

1 Introduction

Ever since Brézis-Vazquez [8] showed that Hardy's inequality can be improved once restricted to a smooth bounded domain Ω in $\mathbb{R}^n$, there was a flurry of activity about possible improvements of the following type:

$$\text{If } n \geq 3 \quad \text{then} \quad \int_\Omega |\nabla u|^2 dx - \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2} dx \geq \int_\Omega V(x)|u|^2 dx \quad \text{for all } u \in H_0^1(\Omega), \tag{1}$$

as well as its fourth order counterpart

$$\text{If } n \geq 5 \quad \text{then} \quad \int_\Omega |\Delta u|^2 dx - \frac{n^2(n-4)^2}{16}\int_\Omega \frac{u^2}{|x|^4} dx \geq \int_\Omega W(x) u^2 dx \quad \text{for } u \in H^2(\Omega) \cap H_0^1(\Omega), \tag{2}$$

where V, W are certain explicit radially symmetric potentials of order lower than $\frac{1}{r^2}$ (for V) and $\frac{1}{r^4}$ (for W). In this paper, we provide an approach that completes, simplifies and improves most related results to-date regarding the Laplacian on Euclidean space as well as its powers. We also establish new inequalities some of which cover critical dimensions such as $n = 2$ for inequality (1) and $n = 4$ for (2).

We start – in section 2 – by giving necessary and sufficient conditions on positive radial functions V and W on a ball B in R^n, so that the following inequality holds for some $c > 0$:

$$\int_B V(x)|\nabla u|^2 dx \geq c\int_B W(x) u^2 dx \text{ for all } u \in C_0^\infty(B). \tag{3}$$

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Assuming that the ball B has radius R and that $\int_0^R \frac{1}{r^{n-1}V(r)}dr = +\infty$, the condition is simply that the ordinary differential equation

$$(\mathrm{B}_{V,cW}) \qquad y''(r) + (\tfrac{n-1}{r} + \tfrac{V_r(r)}{V(r)})y'(r) + \tfrac{cW(r)}{V(r)}y(r) = 0$$

has a positive solution on the interval $(0, R)$. We shall call such a couple (V, W) a *Bessel pair on* $(0, R)$. The *weight* of such a pair is then defined as

$$\beta(V, W; R) = \sup\big\{c;\ (\mathrm{B}_{V,cW}) \text{ has a positive solution on } (0, R)\big\}. \tag{4}$$

This characterization makes an important connection between Hardy-type inequalities and the oscillatory behaviour of the above equations. For example, by using recent results on ordinary differential equations, we can then infer that an integral condition on V, W of the form

$$\limsup_{r\to 0} r^{2(n-1)}V(r)W(r)\Big(\int_r^R \frac{d\tau}{\tau^{n-1}V(\tau)}\Big)^2 < \frac{1}{4} \tag{5}$$

is sufficient (and "almost necessary") for (V, W) to be a Bessel pair on a ball of sufficiently small radius ρ.

Applied in particular, to a pair $(V, \frac{1}{r^2}V)$ where the function $\frac{rV'(r)}{V(r)}$ is assumed to decrease to $-\lambda$ on $(0, R)$, we obtain the following extension of Hardy's inequality: If $\lambda \leq n-2$, then

$$\int_B V(x)|\nabla u|^2 dx \geq (\tfrac{n-\lambda-2}{2})^2 \int_B V(x)\tfrac{u^2}{|x|^2}dx \quad \text{ for all } u \in C_0^\infty(B) \tag{6}$$

and $(\frac{n-\lambda-2}{2})^2$ is the best constant. The case where $V(x) \equiv 1$ is obviously the classical Hardy inequality and when $V(x) = |x|^{-2a}$ for $-\infty < a < \frac{n-2}{2}$, this is a particular case of the Caffarelli-Kohn-Nirenberg inequality. One can however apply the above criterium to obtain new inequalities such as the following: For $a, b > 0$

- If $\alpha\beta > 0$ and $m \leq \frac{n-2}{2}$, then for all $u \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m}}|\nabla u|^2 dx \geq (\frac{n-2m-2}{2})^2 \int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m+2}}u^2 dx, \tag{7}$$

  and $(\frac{n-2m-2}{2})^2$ is the best constant in the inequality.

- If $\alpha\beta < 0$ and $2m - \alpha\beta \leq n-2$, then for all $u \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m}}|\nabla u|^2 dx \geq (\frac{n-2m+\alpha\beta-2}{2})^2 \int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m+2}}u^2 dx, \tag{8}$$

  and $(\frac{n-2m+\alpha\beta-2}{2})^2$ is the best constant in the inequality.

We can also extend some of the recent results of Blanchet-Bonforte-Dolbeault-Grillo-Vasquez [4].

- If $\alpha\beta < 0$ and $-\alpha\beta \leq n-2$, then for all $u \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (a+b|x|^\alpha)^\beta|\nabla u|^2 dx \geq b^{\frac{2}{\alpha}}(\frac{n-\alpha\beta-2}{2})^2 \int_{\mathbb{R}^n} (a+b|x|^\alpha)^{\beta-\frac{2}{\alpha}}u^2 dx, \tag{9}$$

  and $b^{\frac{2}{\alpha}}(\frac{n-\alpha\beta-2}{2})^2$ is the best constant in the inequality.

- If $\alpha\beta > 0$, and $n \geq 2$, then there exists a constant $C > 0$ such that for all $u \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (a+b|x|^\alpha)^\beta|\nabla u|^2 dx \geq C\int_{\mathbb{R}^n} (a+b|x|^\alpha)^{\beta-\frac{2}{\alpha}}u^2 dx. \tag{10}$$

  Moreover, $b^{\frac{2}{\alpha}}(\frac{n-2}{2})^2 \leq C \leq b^{\frac{2}{\alpha}}(\frac{n+\alpha\beta-2}{2})^2$.

On the other hand, by considering the pair

$$V(x) = |x|^{-2a} \quad \text{and} \quad W_{a,c}(x) = (\tfrac{n-2a-2}{2})^2 |x|^{-2a-2} + c|x|^{-2a} W(x)$$

we get the following improvement of the Caffarelli-Kohn-Nirenberg inequalities:

$$\int_B |x|^{-2a}|\nabla u|^2 dx - (\frac{n-2a-2}{2})^2 \int_B |x|^{-2a-2} u^2 dx \geq c \int_B |x|^{-2a} W(x) u^2 dx \quad \text{for all } u \in C_0^\infty(B) \tag{11}$$

if and only if the following ODE

$$(\mathrm{B}_{cW}) \qquad y'' + \tfrac{1}{r} y' + cW(r) y = 0$$

has a positive solution on $(0, R)$. Such a function W will be called a *Bessel potential* on $(0, R)$. This type of characterization was established recently by the authors [15] in the case where $a = 0$, yielding in particular the recent improvements of Hardy's inequalities (on bounded domains) established by Brezis-Vázquez [8], Adimurthi et al. [1], and Filippas-Tertikas [13]. Our results here include in addition those proved by Wang-Willem [27] in the case where $a < \frac{n-2}{2}$ and $W(r) = \frac{1}{r^2 (\ln \frac{R}{r})^2}$, but also cover the previously unknown limiting case corresponding to $a = \frac{n-2}{2}$ as well as the critical dimension $n = 2$.
More importantly, we establish here that Bessel pairs lead to a myriad of optimal Hardy-Rellich inequalities of arbitrary high order, therefore extending and completing a series of new results by Adimurthi et al. [2], Tertikas-Zographopoulos [24] and others. They are mostly based on the following theorem which summarizes the main thrust of this paper.

Theorem 1.1 *Let V and W be positive radial C^1-functions on $B \setminus \{0\}$, where B is a ball centered at zero with radius R in $\mathbb{R}^n$ ($n \geq 1$) such that $\int_0^R \frac{1}{r^{n-1} V(r)} dr = +\infty$ and $\int_0^R r^{n-1} V(r) dr < +\infty$. The following statements are then equivalent:*

1. *(V, W) is a Bessel pair on $(0, R)$ and $\beta(V, W; R) \geq 1$.*
2. *$\int_B V(x) |\nabla u|^2 dx \geq \int_B W(x) u^2 dx$ for all $u \in C_0^\infty(B)$.*
3. *If $\lim_{r \to 0} r^\alpha V(r) = 0$ for some $\alpha < n-2$, then the above are equivalent to*

   $$\int_B V(x) |\Delta u|^2 dx \geq \int_B W(x) |\nabla u|^2 dx + (n-1) \int_B (\tfrac{V(x)}{|x|^2} - \tfrac{V_r(|x|)}{|x|}) |\nabla u|^2 dx \quad \textit{for all radial } u \in C_{0,r}^\infty(B).$$

4. *If in addition, $W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r) \geq 0$ on $(0, R)$, then the above are equivalent to*

   $$\int_B V(x) |\Delta u|^2 dx \geq \int_B W(x) |\nabla u|^2 dx + (n-1) \int_B (\tfrac{V(x)}{|x|^2} - \tfrac{V_r(|x|)}{|x|}) |\nabla u|^2 dx \quad \textit{for all } u \in C_0^\infty(B).$$

In other words, one can obtain as many Hardy and Hardy-Rellich type inequalities as one can construct Bessel pairs on $(0, R)$. The relevance of the above result stems from the fact that there are plenty of such pairs that are easily identifiable. Indeed, even the class of *Bessel potentials* –equivalently those W such that $\big(1, (\frac{n-2}{2})^2 |x|^{-2} + cW(x)\big)$ is a Bessel pair– is quite rich and contains several important potentials. Here are some of the most relevant properties –to be established in an appendix– of the class of C^1 Bessel potentials W on $(0, R)$, that we shall denote by $\mathcal{B}(0, R)$.
First, the class is a closed convex *solid* subset of $C^1(0, R)$, that is if $W \in \mathcal{B}(0, R)$ and $0 \leq V \leq W$, then $V \in \mathcal{B}(0, R)$. The "weight" of each $W \in \mathcal{B}(R)$, that is

$$\beta(W; R) = \sup \big\{ c > 0;\ (B_{cW}) \text{ has a positive solution on } (0, R) \big\}, \tag{12}$$

will be an important ingredient for computing the best constants in corresponding functional inequalities. Here are some basic examples of Bessel potentials and their corresponding weights.

- $W \equiv 0$ is a Bessel potential on $(0, R)$ for any $R > 0$.

- $W \equiv 1$ is a Bessel potential on $(0, R)$ for any $R > 0$, and $\beta(1; R) = \frac{z_0^2}{R^2}$ where $z_0 = 2.4048...$ is the first zero of the Bessel function J_0.

- If $a < 2$, then there exists $R_a > 0$ such that $W(r) = r^{-a}$ is a Bessel potential on $(0, R_a)$.

- For $k \geq 1$, $R > 0$ and $\rho = R(e^{e^{e^{\cdot^{\cdot^{e((k-1)-times)}}}}})$, let $W_{k,\rho}(r) = \Sigma_{j=1}^{k} \frac{1}{r^2} \left(\prod_{i=1}^{j} log^{(i)} \frac{\rho}{r} \right)^{-2}$ where the functions $log^{(i)}$ are defined iteratively as follows: $log^{(1)}(.) = log(.)$ and for $k \geq 2$, $log^{(k)}(.) = log(log^{(k-1)}(.))$. $W_{k,\rho}$ is then a Bessel potential on $(0, R)$ with $\beta(W_{k,\rho}; R) = \frac{1}{4}$.

- For $k \geq 1$, $R > 0$ and $\rho \geq R$, define $\tilde{W}_{k;\rho}(r) = \Sigma_{j=1}^{k} \frac{1}{r^2} X_1^2(\frac{r}{\rho}) X_2^2(\frac{r}{\rho}) \ldots X_{j-1}^2(\frac{r}{\rho}) X_j^2(\frac{r}{\rho})$ where the functions X_i are defined iteratively as follows: $X_1(t) = (1 - \log(t))^{-1}$ and for $k \geq 2$, $X_k(t) = X_1(X_{k-1}(t))$. Then again $\tilde{W}_{k,\rho}$ is a Bessel potential on $(0, R)$ with $\beta(\tilde{W}_{k,\rho}; R) = \frac{1}{4}$.

- More generally, if W is any positive function on $\mathbb{R}$ such that $\liminf_{r \to 0} \ln(r) \int_0^r sW(s)ds > -\infty$, then for every $R > 0$, there exists $\alpha := \alpha(R) > 0$ such that $W_\alpha(x) := \alpha^2 W(\alpha x)$ is a Bessel potential on $(0, R)$.

What is remarkable is that the class of Bessel potentials W is also the one that leads to optimal improvements for fourth order inequalities (in dimension $n \geq 3$) of the following type:

$$\int_B |\Delta u|^2 dx - C(n) \int_B \frac{|\nabla u|^2}{|x|^2} dx \geq c(W, R) \int_B W(x) |\nabla u|^2 dx \quad \text{for all } u \in H_0^2(B), \tag{13}$$

where $C(3) = \frac{25}{36}$, $C(4) = 3$ and $C(n) = \frac{n^2}{4}$ for $n \geq 5$. The case when $W \equiv \tilde{W}_{k,\rho}$ and $n \geq 5$ was recently established by Tertikas-Zographopoulos [24]. Note that W can be chosen to be any one of the examples of Bessel potentials listed above. Moreover, both $C(n)$ and the weight $\beta(W; R)$ are the best constants in the above inequality.

Appropriate combinations of (3) and (13) then lead to a myriad of Hardy-Rellich inequalities in dimension $n \geq 4$. For example, if W is a Bessel potential on $(0, R)$ such that the function $r\frac{W_r(r)}{W(r)}$ decreases to $-\lambda$, and if $\lambda \leq n - 2$, then we have for all $u \in C_0^\infty(B_R)$

$$\int_B |\Delta u|^2 dx - \frac{n^2(n-4)^2}{16} \int_B \frac{u^2}{|x|^4} dx \geq \left(\frac{n^2}{4} + \frac{(n - \lambda - 2)^2}{4}\right) \beta(W; R) \int_B \frac{W(x)}{|x|^2} u^2 dx. \tag{14}$$

By applying (14) to the various examples of Bessel functions listed above, one improves in many ways the recent results of Adimurthi et al. [2] and those by Tertikas-Zographopoulos [24]. Moreover, besides covering the critical dimension $n = 4$, we also establish that the best constant is $(1 + \frac{n(n-4)}{8})$ for all the potentials W_k and $\tilde{W}_k$ defined above. For example we have for $n \geq 4$,

$$\int_B |\Delta u(x)|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_B \frac{u^2}{|x|^4} dx + \left(1 + \frac{n(n-4)}{8}\right) \sum_{j=1}^{k} \int_B \frac{u^2}{|x|^4} \left(\prod_{i=1}^{j} log^{(i)} \frac{\rho}{|x|} \right)^{-2} dx. \tag{15}$$

More generally, we show that for any $m < \frac{n-2}{2}$, and any W Bessel potential on a ball $B_R \subset R^n$ of radius R, the following inequality holds for all $u \in C_0^\infty(B_R)$

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \geq a_{n,m} \int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx + \beta(W; R) \int_{B_R} W(x) \frac{|\nabla u|^2}{|x|^{2m}} dx, \tag{16}$$

where $a_{m,n}$ and $\beta(W; R)$ are best constants that we compute in the appendices for all m and n and for many Bessel potentials W. Worth noting is Corollary 3.2 where we show that inequality (16) restricted to radial functions in $C_0^\infty(B_R)$ holds with a best constant equal to $(\frac{n+2m}{2})^2$, but that $a_{n,m}$ can however be strictly smaller than $(\frac{n+2m}{2})^2$ in the non-radial case. These results improve considerably Theorem 1.7, Theorem 1.8, and Theorem 6.4 in [24].

We also establish a more general version of equation (14). Assuming again that $\frac{rW'(r)}{W(r)}$ decreases to $-\lambda$ on $(0,R)$, and provided $m \le \frac{n-4}{2}$ and $\lambda \le n-2m-2$, we then have for all $u \in C_0^\infty(B_R)$,

$$\begin{aligned}\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}}dx \quad &\ge \quad \beta_{n,m}\int_{B_R}\frac{u^2}{|x|^{2m+4}}dx \\ &+\beta(W;R)\Big(\frac{(n+2m)^2}{4}+\frac{(n-2m-\lambda-2)^2}{4}\Big)\int_{B_R}\frac{W(x)}{|x|^{2m+2}}u^2dx, \qquad (17)\end{aligned}$$

where again the best constants $\beta_{n,m}$ are computed in section 3. This completes the results in Theorem 1.6 of [24], where the inequality is established for $n \ge 5$, $0 \le m < \frac{n-4}{2}$, and the particular potential $\tilde{W}_{k,\rho}$.
Another inequality that relates the Hessian integral to the Dirichlet energy is the following: Assuming $-1 < m \le \frac{n-4}{2}$ and W is a Bessel potential on a ball B of radius R in R^n, then for all $u \in C_0^\infty(B)$,

$$\begin{aligned}\int_B \frac{|\Delta u|^2}{|x|^{2m}}dx - \frac{(n+2m)^2(n-2m-4)^2}{16}\int_B \frac{u^2}{|x|^{2m+4}}dx \quad &\ge \quad \beta(W;R)\frac{(n+2m)^2}{4}\int_B \frac{W(x)}{|x|^{2m+2}}u^2dx \\ &+\beta(|x|^{2m};R)||u||_{H_0^1}. \qquad (18)\end{aligned}$$

This improves considerably Theorem A.2. in [2] where it is established – for $m=0$ and without best constants – with the potential $W_{1,\rho}$ in dimension $n \ge 5$, and the potential $W_{2,\rho}$ when $n=4$.
Finally, we establish several higher order Rellich inequalities for integrals of the form $\int_{B_R}\frac{|\Delta^m u|^2}{|x|^{2k}}dx$, improving in many ways several recent results in [24].
The approach can also be used to improve the recent results of Liskevich-Lyachova-Moroz [18] on exterior domains and will be developed in a forthcoming paper.

2 General Hardy Inequalities

Here is the main result of this section.

Theorem 2.1 *Let V and W be positive radial C^1-functions on $B_R\backslash\{0\}$, where B_R is a ball centered at zero with radius R ($0 < T \le +\infty$) in $\mathbb{R}^n$ ($n \ge 1$). Assume that $\int_0^a \frac{1}{r^{n-1}V(r)}dr = +\infty$ and $\int_0^a r^{n-1}V(r)dr < \infty$ for some $0 < a < R$. Then the following two statements are equivalent:*

1. *The ordinary differential equation*

   $(\mathrm{B}_{V,W})$ $$y''(r) + \Big(\frac{n-1}{r} + \frac{V_r(r)}{V(r)}\Big)y'(r) + \frac{W(r)}{V(r)}y(r) = 0$$

   has a positive solution on the interval $(0,R]$ (possibly with $\varphi(R)=0$).

2. *For all $u \in C_0^\infty(B_R)$*

   $(\mathrm{H}_{V,W})$ $$\int_{B_R} V(x)|\nabla u(x)|^2dx \ge \int_{B_R} W(x)u^2dx.$$

Before proceeding with the proofs, we note the following immediate but useful corollary.

Corollary 2.2 *Let V and W be positive radial C^1-functions on $B\backslash\{0\}$, where B is a ball with radius R in $\mathbb{R}^n$ ($n \ge 1$) and centered at zero, such that $\int_0^R \frac{1}{r^{n-1}V(r)}dr = +\infty$ and $\int_0^R r^{n-1}V(r)dr < \infty$. Then (V,W) is a Bessel pair on $(0,R)$ if and only if for all $u \in C_0^\infty(B_R)$, we have*

$$\int_{B_R} V(x)|\nabla u|^2dx \ge \beta(V,W;R)\int_{B_R} W(x)u^2dx,$$

with $\beta(V,W;R)$ being the best constant.

For the proof of Theorem 2.1, we shall need the following lemmas.

Lemma 2.3 *Let Ω be a smooth bounded domain in R^n with $n \geq 1$ and let $\varphi \in C^1(0, R := \sup_{x\in\partial\Omega}|x|)$ be a positive solution of the ordinary differential equation*

$$y'' + (\frac{n-1}{r} + \frac{V_r(r)}{V(r)})y' + \frac{W(r)}{V(r)}y = 0, \tag{19}$$

on $(0,R)$ for some $V(r), W(r) \geq 0$ where $\int_0^R \frac{1}{r^{n-1}V(r)}dr = +\infty$ and $\int_0^R r^{n-1}V(r)dr < \infty$. Setting $\psi(x) = \frac{u(x)}{\varphi(|x|)}$ for any $u \in C_0^\infty(\Omega)$, we then have the following properties:

1. $\int_0^R r^{n-1}V(r)(\frac{\varphi'(r)}{\varphi(r)})^2dr < \infty$ *and* $\lim_{r\to 0} r^{n-1}V(r)\frac{\varphi'(r)}{\varphi(r)} = 0$.
2. $\int_\Omega V(|x|)(\varphi'(|x|))^2\psi^2(x)dx < \infty$.
3. $\int_\Omega V(|x|)\varphi^2(|x|)|\nabla\psi|^2(x)dx < \infty$.
4. $|\int_\Omega V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi(x)dx| < \infty$.
5. $\lim_{r\to 0}|\int_{\partial B_r} V(|x|)\varphi'(|x|)\varphi(|x|)\psi^2(x)ds| = 0$, *where $B_r \subset \Omega$ is a ball of radius r centered at 0.*

Proof: 1) Setting $x(r) = r^{n-1}V(r)\frac{\varphi'(r)}{\varphi(r)}$, we have

$$r^{n-1}V(r)x'(r)+x^2(r) = \frac{r^{2(n-1)}V^2(r)}{\varphi}(\varphi''(r)+(\frac{n-1}{r}+\frac{V_r(r)}{V(r)})\varphi'(r)) = -\frac{r^{2(n-1)}V(r)W(r)}{\varphi(r)} \leq 0, \quad 0<r<R.$$

Dividing by $r^{n-1}V(r)$ and integrating once, we obtain

$$x(r) \geq \int_r^R \frac{|x(s)|^2}{s^{n-1}V(s)}ds + x(R). \tag{20}$$

To prove that $\lim_{r\to 0} G(r) < \infty$, where $G(r) := \int_r^R \frac{x^2(s)}{s^{n-1}V(s)}ds$, we assume the contrary and use (20) to write that

$$(-r^{n-1}V(r))G'(r))^{\frac{1}{2}} \geq G(r) + x(R).$$

Thus, for r sufficiently small we have $-r^{n-1}V(r)G'(r) \geq \frac{1}{2}G^2(r)$ and hence, $(\frac{1}{G(r)})' \geq \frac{1}{2r^{n-1}V(r)}$, which contradicts the fact that $G(r)$ goes to infinity as r tends to zero.
Also in view of (20), we have that $x_0 := \lim_{r\to 0} x(r)$ exists, and since $\lim_{r\to 0} G(r) < \infty$, we necessarily have $x_0 = 0$ and 1) is proved.
For assertion 2), we use 1) to see that

$$\int_\Omega V(|x|)(\varphi'(|x|))^2\psi^2(x)dx \leq ||u||_\infty^2 \int_\Omega V(|x|)\frac{(\varphi'(|x|))^2}{\varphi^2(|x|)}dx < \infty.$$

3) Note that

$$|\nabla\psi(x)| \leq \frac{|\nabla u(x)|}{\varphi(|x|)} + |u(x)|\frac{|\varphi'(|x|)|}{\varphi^2(|x|)} \leq \frac{C_1}{\varphi(|x|)} + C_2\frac{|\varphi'(|x|)|}{\varphi^2(|x|)}, \quad \text{for all } x \in \Omega,$$

where $C_1 = \max_{x\in\Omega}|\nabla u|$ and $C_2 = \max_{x\in\Omega}|u|$. Hence we have

$$\begin{aligned}
\int_\Omega V(|x|)\varphi^2(|x|)|\nabla\psi|^2(x)dx &\leq \int_\omega V(|x|)\frac{(C_1\varphi(|x|)+C_2\varphi'(|x|))^2}{\varphi^2(|x|)}dx \\
&= \int_\Omega C_1^2V(|x|)dx + \int_\Omega 2C_1C_2\frac{|\varphi'(|x|)|}{\varphi(|x|)}V(|x|)dx + \int_\Omega C_2^2(\frac{\varphi'(|x|)}{\varphi(|x|)})^2V(|x|)dx \\
&\leq L_1 + 2C_1C_2\Big(\int_\Omega V(|x|)(\frac{\varphi'(|x|)}{\varphi(|x|)})^2dx\Big)^{\frac{1}{2}}\Big(\int_\Omega V(|x|)dx\Big)^{\frac{1}{2}} + L_2 \\
&< \infty,
\end{aligned}$$

which proves 3).
4) now follows from 2) and 3) since

$$V(|x|)|\nabla u|^2 = V(|x|)(\varphi'(|x|))^2\psi^2(x) + 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi(x) + V(|x|)\varphi^2(|x|)|\nabla\psi|^2.$$

Finally, 5) follows from 1) since

$$\begin{aligned}|\int_{\partial B_r} V(|x|)\varphi'(|x|)\varphi(|x|)\psi^2(x)ds| &< ||u||_\infty^2|\int_{\partial B_r} V(|x|)\frac{\varphi'(|x|)}{\varphi(|x|)}ds\\ &= ||u||_\infty^2 V(r)\frac{|\varphi'(r)|}{\varphi(r)}\int_{\partial B_r} 1ds\\ &= n\omega_n||u||_\infty^2 r^{n-1}V(r)\frac{|\varphi'(r)|}{\varphi(r)}.\end{aligned}$$

Lemma 2.4 *Let V and W be positive radial C^1-functions on a ball $B\backslash\{0\}$, where B is a ball with radius R in $\mathbb{R}^n$ ($n \geq 1$) and centered at zero. Assuming*

$$\textstyle\int_B \left(V(x)|\nabla u|^2 - W(x)|u|^2\right)dx \geq 0 \text{ for all } u \in C_0^\infty(B),$$

then there exists a C^2-supersolution to the following linear elliptic equation

$$-\operatorname{div}(V(x)\nabla u) - W(x)u = 0, \quad \text{in } B, \tag{21}$$
$$u > 0 \quad \text{in } B\setminus\{0\}, \tag{22}$$
$$u = 0 \quad \text{in } \partial B. \tag{23}$$

Proof: Define

$$\lambda_1(V) := \inf\{\frac{\int_B V(x)|\nabla\psi|^2 - W(x)|\psi|^2}{\int_B |\psi|^2};\quad \psi \in C_0^\infty(B\setminus\{0\})\}.$$

By our assumption $\lambda_1(V) \geq 0$. Let (φ_n, λ_1^n) be the first eigenpair for the problem

$$\begin{aligned}(L - \lambda_1(V) - \lambda_1^n)\varphi_n &= 0 \text{ on } B\setminus B_{\frac{R}{n}}\\ \varphi_n &= 0 \text{ on } \partial(B\setminus B_{\frac{R}{n}}),\end{aligned}$$

where $Lu = -\operatorname{div}(V(x)\nabla u) - W(x)u$, and $B_{\frac{R}{n}}$ is a ball of radius $\frac{R}{n}$, $n \geq 2$. The eigenfunctions can be chosen in such a way that $\varphi_n > 0$ on $B\setminus B_{\frac{R}{n}}$ and $\varphi_n(b) = 1$, for some $b \in B$ with $\frac{R}{2} < |b| < R$.
Note that $\lambda_1^n \downarrow 0$ as $n \to \infty$. Harnak's inequality yields that for any compact subset K, $\frac{\max_K \varphi_n}{\min_K \varphi_n} \leq C(K)$ with the later constant being independant of φ_n. Also standard elliptic estimates also yields that the family (φ_n) have also uniformly bounded derivatives on the compact sets $B - B_{\frac{R}{n}}$.
Therefore, there exists a subsequence $(\varphi_{n_{l_2}})_{l_2}$ of $(\varphi_n)_n$ such that $(\varphi_{n_{l_2}})_{l_2}$ converges to some $\varphi_2 \in C^2(B\setminus B(\frac{R}{2}))$. Now consider $(\varphi_{n_{l_2}})_{l_2}$ on $B\setminus B(\frac{R}{3})$. Again there exists a subsequence $(\varphi_{n_{l_3}})_{l_3}$ of $(\varphi_{n_{l_2}})_{l_2}$ which converges to $\varphi_3 \in C^2(B\setminus B(\frac{R}{3}))$, and $\varphi_3(x) = \varphi_2(x)$ for all $x \in B\setminus B(\frac{R}{2})$. By repeating this argument we get a supersolution $\varphi \in C^2(B\setminus\{0\})$ i.e. $L\varphi \geq 0$, such that $\varphi > 0$ on $B\setminus\{0\}$. □

Proof of Theorem 2.1: First we prove that 1) implies 2). Let $\varphi \in C^1(0,R]$ be a solution of $(B_{V,W})$ such that $\varphi(x) > 0$ for all $x \in (0,R)$. Define $\frac{u(x)}{\varphi(|x|)} = \psi(x)$. Then

$$|\nabla u|^2 = (\varphi'(|x|))^2\psi^2(x) + 2\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi + \varphi^2(|x|)|\nabla\psi|^2.$$

Hence,

$$V(|x|)|\nabla u|^2 \geq V(|x|)(\varphi'(|x|))^2\psi^2(x) + 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi(x).$$

Thus, we have

$$\int_B V(|x|)|\nabla u|^2 dx \geq \int_B V(|x|)(\varphi'(|x|))^2\psi^2(x)dx + \int_B 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi dx.$$

Let B_ϵ be a ball of radius ϵ centered at the origin. Integrate by parts to get

$$\begin{aligned}
\int_B V(|x|)|\nabla u|^2 dx &\geq \int_B V(|x|)(\varphi'(|x|))^2\psi^2(x)dx + \int_{B_\epsilon} 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi dx \\
&+ \int_{B\setminus B_\epsilon} 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi dx \\
&= \int_{B_\epsilon} V(|x|)(\varphi'(|x|))^2\psi^2(x)dx + \int_{B_\epsilon} 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi dx \\
&- \int_{B\setminus B_\epsilon} \left\{\big(V(|x|)\varphi''(|x|)\varphi(|x|) + (\frac{(n-1)V(|x|)}{r} + V_r(|x|))\varphi'(|x|)\varphi(|x|)\big)\psi^2(x)\right\} dx \\
&+ \int_{\partial(B\setminus B_\epsilon)} V(|x|)\varphi'(|x|)\varphi(|x|)\psi^2(x)ds
\end{aligned}$$

Let $\epsilon \to 0$ and use Lemma 2.3 and the fact that φ is a solution of $(D_{v,w})$ to get

$$\begin{aligned}
\int_B V(|x|)|\nabla u|^2 dx &\geq -\int_B [V(|x|)\varphi''(|x|) + (\frac{(n-1)V(|x|)}{r} + V_r(|x|))\varphi'(|x|)]\frac{u^2(x)}{\varphi(|x|)}dx \\
&= \int_B W(|x|)u^2(x)dx.
\end{aligned}$$

To show that 2) implies 1), we assume that inequality ($\mathrm{H}_{V,W}$) holds on a ball B of radius R, and then apply Lemma 2.4 to obtain a C^2-supersolution for the equation (21). Now take the surface average of u, that is

$$y(r) = \frac{1}{n\omega_w r^{n-1}}\int_{\partial B_r} u(x)dS = \frac{1}{n\omega_n}\int_{|\omega|=1} u(r\omega)d\omega > 0, \tag{24}$$

where ω_n denotes the volume of the unit ball in R^n. We may assume that the unit ball is contained in B (otherwise we just use a smaller ball). We clearly have

$$y''(r) + \frac{n-1}{r}y'(r) = \frac{1}{n\omega_n r^{n-1}}\int_{\partial B_r} \Delta u(x)dS. \tag{25}$$

Since $u(x)$ is a supersolution of (21), we have

$$\int_{\partial B_r} div(V(|x|)\nabla u)ds - \int_{\partial B} W(|x|)udx \geq 0,$$

and therefore,

$$V(r)\int_{\partial B_r} \Delta u dS - V_r(r)\int_{\partial B_r} \nabla u.x ds - W(r)\int_{\partial B_r} u(x)ds \geq 0.$$

It follows that

$$V(r)\int_{\partial B_r} \Delta u dS - V_r(r)y'(r) - W(r)y(r) \geq 0, \tag{26}$$

and in view of (24), we see that y satisfies the inequality

$$V(r)y''(r) + (\frac{(n-1)V(r)}{r} + V_r(r))y'(r) \leq -W(r)y(r), \quad for \quad 0 < r < R, \tag{27}$$

that is it is a positive supersolution for $(B_{V,W})$.
Standard results in ODE now allow us to conclude that $(B_{V,W})$ has actually a positive solution on $(0, R)$, and the proof of theorem 2.1 is now complete.

2.1 Integral criteria for Bessel pairs

In order to obtain criteria on V and W so that inequality ($\mathrm{H}_{V,W}$) holds, we clearly need to investigate whether the ordinary differential equation ($B_{V,W}$) has positive solutions. For that, we rewrite ($B_{V,W}$) as

$$(r^{n-1}V(r)y')' + r^{n-1}W(r)y = 0,$$

and then by setting $s = \frac{1}{r}$ and $x(s) = y(r)$, we see that y is a solution of ($B_{V,W}$) on an interval $(0, \delta)$ if and only if x is a positive solution for the equation

$$(s^{-(n-3)}V(\tfrac{1}{s})x'(s))' + s^{-(n+1)}W(\tfrac{1}{s})x(s) = 0 \quad \text{on} \quad (\tfrac{1}{\delta}, \infty). \tag{28}$$

Now recall that a solution $x(s)$ of the equation (28) is said to be *oscillatory* if there exists a sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n \to +\infty$ and $x(a_n) = 0$. Otherwise we call the solution *non-oscillatory*. It follows from Sturm comparison theorem that all solutions of (28) are either all oscillatory or all non-oscillatory. Hence, the fact that (V, W) is a Bessel pair or not is closely related to the oscillatory behavior of the equation (28). The following theorem is therefore a consequence of Theorem 2.1, combined with a relatively recent result of Sugie et al. in [22] about the oscillatory behavior of the equation (28).

Theorem 2.5 *Let V and W be positive radial C^1-functions on $B_R \backslash \{0\}$, where B_R is a ball centered at 0 with radius R in $\mathbb{R}^n$ $(n \geq 1)$. Assume $\int_0^R \frac{1}{\tau^{n-1}V(\tau)} d\tau = +\infty$ and $\int_0^R r^{n-1}v(r)dr < \infty$.*

- *Assume*
$$\limsup_{r \to 0} r^{2(n-1)}V(r)W(r)\Big(\int_r^R \frac{1}{\tau^{n-1}V(\tau)} d\tau\Big)^2 < \frac{1}{4} \tag{29}$$
then (V, W) is a Bessel pair on $(0, \rho)$ for some $\rho > 0$ and consequently, inequality ($\mathrm{H}_{V,W}$) holds for all $u \in C_0^\infty(B_\rho)$, where B_ρ is a ball of radius ρ.

- *On the other hand, if*
$$\liminf_{r \to 0} r^{2(n-1)}V(r)W(r)\Big(\int_r^R \frac{1}{\tau^{n-1}V(\tau)} d\tau\Big)^2 > \frac{1}{4} \tag{30}$$
then there is no interval $(0, \rho)$ on which (V, W) is a Bessel pair and consequently, there is no smooth domain Ω on which inequality ($\mathrm{H}_{V,W}$) holds.

A typical Bessel pair is $(|x|^{-\lambda}, |x|^{-\lambda-2})$ for $\lambda \leq n-2$. It is also easy to see by a simple change of variables in the corresponding ODEs that

$$W \text{ is a Bessel potential if and only if } \big(|x|^{-\lambda}, |x|^{-\lambda}(|x|^{-2} + W(|x|)\big) \text{ is a Bessel pair.} \tag{31}$$

More generally, the above integral criterium allows to show the following.

Theorem 2.6 *Let V be an strictly positive C^1-function on $(0, R)$ such that for some $\lambda \in \mathbb{R}$*

$$\tfrac{rV_r(r)}{V(r)} + \lambda \geq 0 \ \textit{on} \ (0, R) \ \textit{and} \ \lim_{r \to 0} \tfrac{rV_r(r)}{V(r)} + \lambda = 0. \tag{32}$$

If $\lambda \leq n-2$, then for any Bessel potential W on $(0, R)$, and any $c \leq \beta(W; R)$, the couple $(V, W_{\lambda,c})$ is a Bessel pair, where

$$W_{\lambda,c}(r) = V(r)((\frac{n-\lambda-2}{2})^2 r^{-2} + cW(r)). \tag{33}$$

Moreover, $\beta\big(V, W_{\lambda,c}; R\big) = 1$ for all $c \leq \beta(W; R)$.

We need the following easy lemma.

Lemma 2.7 *Assume the equation*

$$y'' + \frac{a}{r}y' + V(r)y = 0,$$

has a positive solution on $(0, R)$, where $a \geq 1$ and $V(r) > 0$. Then y is strictly decreasing on $(0, R)$.

Proof: First observe that y can not have a local minimum, hence it is either increasing or decreasing on $(0,\delta)$, for δ sufficiently small. Assume y is increasing. Under this assumption if $y'(a)=0$ for some $a>0$, then $y''(a)=0$ which contradicts the fact that y is a positive solution of the above ODE. So we have $\frac{y''}{y'}\leq -\frac{a}{r}$, thus,

$$y'\geq \frac{c}{r^a}.$$

Therefore, $x(r)\to -\infty$ as $r\to 0$ which is a contradiction. Since, y can not have a local minimum it should be strictly decreasing on $(0,R)$. □

Proof of Theorem 2.6: Write $\frac{V_r(r)}{V(r)}=-\frac{\lambda}{r}+f(r)$ where $f(r)\geq 0$ on $(0,R)$ and $\lim_{r\to 0} rf(r)=0$. In order to prove that $\big(V(r),V(r)((\frac{n-\lambda-2}{2})^2r^{-2}+cW(r))\big)$ is a Bessel pair, we need to show that the equation

$$y''+(\frac{n-\lambda-1}{r}+f(r))y'+((\frac{n-\lambda-2}{2})^2r^{-2}+cW(r))y(r)=0, \tag{34}$$

has a positive solution on $(0,R)$. But first we note that the equation

$$x''+(\frac{n-\lambda-1}{r})x'+((\frac{n-\lambda-2}{2})^2r^{-2}+cW(r))x(r)=0,$$

has a positive solution on $(0,R)$, whenever $c\leq\beta(W;R)$. Since now $f(r)\geq 0$ and since, by the proceeding lemma, $x'(r)\leq 0$, we get that x is a positive subsolution for the equation (34) on $(0,R)$, and thus it has a positive solution of $(0,R)$. Note that this means that $\beta(V,W_{\lambda,c};R)\geq 1$.
For the reverse inequality, we shall use the criterium in Theorem 2.5. Indeed apply criteria (29) to $V(r)$ and $W_1(r)=C\frac{V(r)}{r^2}$ to get

$$\begin{aligned}
\lim_{r\to 0} r^{2(n-1)}V(r)W_1(r)\Big(\int_r^R\frac{1}{\tau^{n-1}V(\tau)}d\tau\Big)^2 &= C\lim_{r\to 0} r^{2(n-2)}V^2(r)\Big(\int_r^R\frac{1}{\tau^{n-1}V(\tau)}d\tau\Big)^2\\
&= C\Big(\lim_{r\to 0} r^{(n-2)}V(r)\int_r^R\frac{1}{\tau^{n-1}V(\tau)}d\tau\Big)^2\\
&= C\Big(\lim_{r\to 0}\frac{\frac{1}{r^{n-1}V(r)}}{\frac{(n-2)r^{n-3}V(r)+r^{n-2}V_r(r)}{r^{2(n-2)}V^2(r)}}\Big)^2\\
&= C\Big(\lim_{r\to 0}\frac{1}{(n-2)+r\frac{V_r(r)}{V(r)}}\Big)^2\\
&= \frac{C}{(n-\lambda-2)^2}.
\end{aligned}$$

For $\big(V,CV(r^{-2}+cW)\big)$ to be a Bessel pair, it is necessary that $\frac{C}{(n-\lambda-2)^2}\leq\frac{1}{4}$, and the proof for the best constant is complete. □
With a similar argument one can also prove the following.

Corollary 2.8 *Let V and W be positive radial C^1-functions on $B_R\backslash\{0\}$, where B_R is a ball centered at zero with radius R in $\mathbb{R}^n$ $(n\geq 1)$. Assume that*

$$\lim_{r\to 0} r\frac{V_r(r)}{V(r)}=-\lambda \textit{ and } \lambda\leq n-2. \tag{35}$$

- *If $\limsup_{r\to 0} r^2\frac{W(r)}{V(r)}<(\frac{n-\lambda-2}{2})^2$, then (V,W) is a Bessel pair on some interval $(0,\rho)$, and consequently there exists a ball $B_\rho\subset R^n$ such that inequality $(\mathrm{H}_{V,W})$ holds for all $u\in C_0^\infty(B_\rho)$.*
- *On the other hand, if $\liminf_{r\to 0} r^2\frac{W(r)}{V(r)}>(\frac{n-\lambda-2}{2})^2$, then there is no smooth domain $\Omega\subset R^n$ such that inequality $(\mathrm{H}_{V,W})$ holds on Ω.*

2.2 New weighted Hardy inequalities

An immediate application of Theorem 2.6 and Theorem 2.1 is the following very general Hardy inequality.

Theorem 2.9 *Let $V(x) = V(|x|)$ be a strictly positive radial function on a smooth domain Ω containing 0 such that $R = \sup_{x\in\Omega} |x|$. Assume that for some $\lambda \in \mathbb{R}$*

$$\frac{rV_r(r)}{V(r)} + \lambda \geq 0 \text{ on } (0,R) \text{ and } \lim_{r\to 0} \frac{rV_r(r)}{V(r)} + \lambda = 0. \tag{36}$$

1. *If $\lambda \leq n-2$, then the following inequality holds for any Bessel potential W on $(0,R)$:*

$$\int_\Omega V(x)|\nabla u|^2 dx \geq (\tfrac{n-\lambda-2}{2})^2 \int_\Omega \tfrac{V(x)}{|x|^2}u^2 dx + \beta(W;R)\int_\Omega V(x)W(x)u^2 dx \quad \text{for all } u \in C_0^\infty(\Omega), \tag{37}$$

and both $(\frac{n-\lambda-2}{2})^2$ and $\beta(W;R)$ are the best constants.

2. *In particular, $\beta(V, r^{-2}V; R) = (\frac{n-\lambda-2}{2})^2$ is the best constant in the following inequality*

$$\int_\Omega V(x)|\nabla u|^2 dx \geq (\tfrac{n-\lambda-2}{2})^2 \int_\Omega \tfrac{V(x)}{|x|^2}u^2 dx \quad \text{for all } u \in C_0^\infty(\Omega). \tag{38}$$

Applied to $V_1(r) = r^{-m}W_{k,\rho}(r)$ and $V_2(r) = r^{-m}\tilde{W}_{k,\rho}(r)$ where $W_{k,\rho}(r) = \Sigma_{j=1}^k \frac{1}{r^2}\big(\prod_{i=1}^j log^{(i)}\frac{\rho}{r}\big)^{-2}$ and $\tilde{W}_{k;\rho}(r) = \Sigma_{j=1}^k \frac{1}{r^2}X_1^2(\frac{r}{\rho})X_2^2(\frac{r}{\rho})\dots X_{j-1}^2(\frac{r}{\rho})X_j^2(\frac{r}{\rho})$ are the iterated logs introduced in the introduction, and noting that in both cases the corresponding λ is equal to $2m+2$, we get the following new Hardy inequalities.

Corollary 2.10 *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ and $m \leq \frac{n-4}{2}$. Then the following inequalities hold.*

$$\int_\Omega \frac{W_{k,\rho}(x)}{|x|^{2m}}|\nabla u|^2 dx \geq (\frac{n-2m-4}{2})^2 \int_\Omega \frac{W_{k,\rho}(x)}{|x|^{2m+2}}u^2 dx \tag{39}$$

$$\int_\Omega \frac{\tilde{W}_{k,\rho}(x)}{|x|^{2m}}|\nabla u|^2 dx \geq (\frac{n-2m-4}{2})^2 \int_\Omega \frac{\tilde{W}_{k,\rho}(x)}{|x|^{2m+2}}u^2 dx. \tag{40}$$

Moreover, the constant $(\frac{n-2m-4}{2})^2$ is the best constant in both inequalities.

Remark 2.11 The two following theorems deal with Hardy-type inequalities on the whole of $\mathbb{R}^n$. Theorem 2.1 already yields that inequality $(H_{V,W})$ holds for all $u \in C_0^\infty(\mathbb{R}^n)$ if and only if the ODE $(B_{V,W})$ has a positive solution on $(0,\infty)$. The latter equation is therefore non-oscillatory, which will again be a very useful fact for computing best constants, in view of the following criterium at infinity (Theorem 2.1 in [22]) applied to the equation

$$(a(r)y')' + b(r)y(r) = 0, \tag{41}$$

where $a(r)$ and $b(r)$ are positive real valued functions. Assuming that $\int_d^\infty \frac{1}{a(\tau)}d\tau < \infty$ for some $d > 0$, and that the following limit

$$L := \lim_{r\to\infty} a(r)b(r)\left(\int_r^\infty \frac{1}{a(r)}dr\right)^2,$$

exists. Then for the equation (41) equation to be non-oscillatory, it is necessary that $L \leq \frac{1}{4}$.

Theorem 2.12 *Let $a, b > 0$, and α, β, m be real numbers.*

- *If $\alpha\beta > 0$, and $m \leq \frac{n-2}{2}$, then for all $u \in C_0^\infty(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m}}|\nabla u|^2 dx \geq (\frac{n-2m-2}{2})^2 \int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m+2}}u^2 dx, \tag{42}$$

and $(\frac{n-2m-2}{2})^2$ is the best constant in the inequality.

- *If $\alpha\beta < 0$, and $2m - \alpha\beta \le n-2$, then for all $u \in C_0^\infty(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m}}|\nabla u|^2 dx \ge (\frac{n-2m+\alpha\beta-2}{2})^2 \int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m+2}} u^2 dx, \tag{43}$$

  and $(\frac{n-2m+\alpha\beta-2}{2})^2$ is the best constant in the inequality.

Proof: Letting $V(r) = \frac{(a+br^\alpha)^\beta}{r^{2m}}$, then

$$r\frac{V'(r)}{V(r)} = -2m + \frac{b\alpha\beta r^\alpha}{a+br^\alpha} = -2m + \alpha\beta - \frac{a\alpha\beta}{a+br^\alpha}.$$

Hence, in the case $\alpha, \beta > 0$ and $2m \le n-2$, (42) follows directly from Theorem 2.9. The same holds for (43) since it also follows directly from Theorem 2.9 in the case where $\alpha < 0$, $\beta > 0$ and $2m - \alpha\beta \le n-2$. For the remaining two other cases, we will use Theorem 2.1. Indeed, in this case the equation $(B_{V,W})$ becomes

$$y'' + (\frac{n-2m-1}{r} + \frac{b\alpha\beta r^{\alpha-1}}{a+br^\alpha})y' + \frac{c}{r^2}y = 0, \tag{44}$$

and the best constant in inequalities (42) and (43) is the largest c such that the above equation has a positive solution on $(0, +\infty)$. Note that by Lemma 2.7, we have that $y' < 0$ on $(0, +\infty)$. Hence, if $\alpha < 0$ and $\beta < 0$, then the positive solution of the equation

$$y'' + \frac{n-2m-1}{r}y' + \frac{(\frac{n-2m-2}{2})^2}{r^2}y = 0$$

is a positive super-solution for (44) and therefore the latter ODE has a positive solution on $(0, +\infty)$, from which we conclude that (42) holds. To prove now that $(\frac{n-2m-2}{2})^2$ is the best constant in (42), we use the fact that if the equation (44) has a positive solution on $(0, +\infty)$, then the equation is necessarily non-oscillatory. By rewriting (44) as

$$\left(r^{n-2m-1}(a+br^\alpha)^\beta y'\right)' + cr^{n-2m-3}(a+br^\alpha)^\beta y = 0, \tag{45}$$

and by noting that

$$\int_d^\infty \frac{1}{r^{n-2m-1}(a+br^\alpha)^\beta} < \infty,$$

and

$$\lim_{r\to\infty} cr^{2(n-2m-2)}(a+br^\alpha)^{2\beta}\left(\int_r^\infty \frac{1}{r^{n-2m-1}(a+br^\alpha)^\beta}dr\right)^2 = \frac{c}{(n-2m-2)^2},$$

we can use Theorem 2.1 in [22] to conclude that for equation (45) to be non-oscillatory it is necessary that

$$\frac{c}{(n-2m-2)^2} \le \frac{1}{4}.$$

Thus, $\frac{(n-2m-2)^2}{4}$ is the best constant in the inequality (42).
A very similar argument applies in the case where $\alpha > 0$, $\beta < 0$, and $2m < n-2$, to obtain that inequality (43) holds for all $u \in C_0^\infty(\mathbb{R}^n)$ and that $(\frac{n-2m+\alpha\beta-2}{2})^2$ is indeed the best constant. □

Note that the above two inequalities can be improved on smooth bounded domains by using Theorem 2.9. We shall now extend the recent results of Blanchet-Bonforte-Dolbeault-Grillo-Vasquez [4] and address some of their questions regarding best constants.

Theorem 2.13 *Let $a, b > 0$, and α, β be real numbers.*

- *If $\alpha\beta < 0$ and $-\alpha\beta \le n-2$, then for all $u \in C_0^\infty(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} (a+b|x|^\alpha)^\beta |\nabla u|^2 dx \ge b^{\frac{2}{\alpha}}(\frac{n-\alpha\beta-2}{2})^2 \int_{\mathbb{R}^n} (a+b|x|^\alpha)^{\beta-\frac{2}{\alpha}} u^2 dx, \tag{46}$$

  and $b^{\frac{2}{\alpha}}(\frac{n-\alpha\beta-2}{2})^2$ is the best constant in the inequality.

- *If $\alpha\beta > 0$ and $n \geq 2$, then there exists a constant $C > 0$ such that for all $u \in C_0^\infty(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} (a + b|x|^\alpha)^\beta |\nabla u|^2 dx \geq C \int_{\mathbb{R}^n} (a + b|x|^\alpha)^{\beta - \frac{2}{\alpha}} u^2 dx. \tag{47}$$

Moreover, $b^{\frac{2}{\alpha}}(\frac{n-2}{2})^2 \leq C \leq b^{\frac{2}{\alpha}}(\frac{n+\alpha\beta-2}{2})^2$.

Proof: Letting $V(r) = (a + br^\alpha)^\beta$, then we have

$$r\frac{V'(r)}{V(r)} = \frac{b\alpha\beta r^\alpha}{a + br^\alpha} = \alpha\beta - \frac{a\alpha\beta}{a + br^\alpha}.$$

Inequality (46) and its best constant in the case when $\alpha < 0$ and $\beta > 0$, then follow immediately from Theorem 2.9 with $\lambda = -\alpha\beta$. The proof of the remaining cases will use Theorem 2.1 as well as the integral criteria for the oscillatory behavior of solutions for ODEs of the form $(B_{V,W})$.
Assuming still that $\alpha\beta < 0$, then with an argument similar to that of Theorem 2.12 above, one can show that the positive solution of the equation $y'' + (\frac{n+\alpha\beta-1}{r})y' + \frac{(n+\alpha\beta-2)^2}{4r^2}y = 0$ on $(0, +\infty)$ is a positive supersolution for the equation

$$y'' + (\frac{n-1}{r} + \frac{V'(r)}{V(r)})y' + \frac{b^{\frac{2}{\alpha}}(n + \alpha\beta - 2)^2}{4(a + br^\alpha)^{\frac{2}{\alpha}}}y = 0.$$

Theorem 2.1 then yields that the inequality (46) holds for all $u \in C_0^\infty(\mathbb{R}^n)$. To prove now that $b^{\frac{2}{\alpha}}(\frac{n+\alpha\beta-2}{2})^2$ is the best constant in (46) it is enough to show that if the following equation

$$\left(r^{n-1}(a + br^\alpha)^\beta y'\right)' + cr^{n-1}(a + br^\alpha)^{\beta - \frac{2}{\alpha}} y = 0 \tag{48}$$

has a positive solution on $(0, +\infty)$, then $c \leq b^{\frac{2}{\alpha}}(\frac{n+\alpha\beta-2}{2})^2$. If now $\alpha > 0$ and $\beta < 0$, then we have

$$\lim_{r\to\infty} cr^{2(n-1)}(a + br^\alpha)^{2\beta - \frac{2}{\alpha}} \left(\int_r^\infty \frac{1}{r^{n-1}(a + br^\alpha)^\beta} dr\right)^2 = \frac{c}{b^{\frac{2}{\alpha}}(n + \alpha\beta - 2)^2}.$$

Hence, by Theorem 2.1 in [22] again, the non-oscillatory aspect of the equation holds for $c \leq \frac{b^{\frac{2}{\alpha}}(n+\alpha\beta-2)^2}{4}$ which completes the proof of the first part.
A similar argument applies in the case where $\alpha\beta > 0$ to prove that (47) holds for all $u \in C_0^\infty(\mathbb{R}^n)$ and $b^{\frac{2}{\alpha}}(\frac{n-2}{2})^2 \leq C \leq b^{\frac{2}{\alpha}}(\frac{n+\alpha\beta-2}{2})^2$. The best constants are estimated by carefully studying the existence of positive solutions for the ODE (48).

Remark 2.14 Recently, Blanchet et al. in [4] studied a special case of inequality (46) ($a = b = 1$, and $\alpha = 2$) under the additional condition:

$$\int_{\mathbb{R}^n} (1 + |x|^2)^{\beta - 1} u(x) dx = 0, \quad for \quad \beta < \frac{n-2}{2}. \tag{49}$$

Note that we do not assume (49) in Theorem 2.13, and that we have found the best constants for $\beta \leq 0$, a case that was left open in [4].

2.3 Improved Hardy and Caffarelli-Kohn-Nirenberg Inequalities

In [9] Caffarelli-Kohn-Nirenberg established a set inequalities of the following form:

$$\left(\int_{R^n} |x|^{-bp}|u|^p dx\right)^{\frac{2}{p}} \leq C_{a,b} \int_{R^n} |x|^{-2a}|\nabla u|^2 dx \text{ for all } u \in C_0^\infty(R^n), \tag{50}$$

where for $n \geq 3$,

$$-\infty < a < \tfrac{n-2}{2},\ a \leq b \leq a + 1, \text{ and } p = \tfrac{2n}{n-2+2(b-a)}. \tag{51}$$

For the cases $n = 2$ and $n = 1$ the conditions are slightly different. For $n = 2$

$$-\infty < a < 0,\ a < b \leq a+1, \text{ and } p = \tfrac{2}{b-a}, \tag{52}$$

and for $n = 1$

$$-\infty < a < -\tfrac{1}{2},\ a + \tfrac{1}{2} < b \leq a+1, \text{ and } p = \tfrac{2}{-1+2(b-a)}. \tag{53}$$

Let $D_a^{1,2}$ be the completion of $C_0^\infty(R^n)$ for the inner product $(u,v) = \int_{R^n} |x|^{-2a} \nabla u . \nabla v dx$ and let

$$S(a,b) = \inf_{u \in D_a^{1,2} \setminus \{0\}} \frac{\int_{R^n} |x|^{-2a} |\nabla u|^2 dx}{\left(\int_{R^n} |x|^{-bp} |u|^p dx\right)^{2/p}} \tag{54}$$

denote the best embedding constant. We are concerned here with the "Hardy critical" case of the above inequalities, that is when $b = a + 1$. In this direction, Catrina and Wang [11] showed that for $n \geq 3$ we have $S(a, a+1) = (\frac{n-2a-2}{2})^2$ and that $S(a, a+1)$ is not achieved while $S(a,b)$ is always achieved for $a < b < a+1$. For the case $n = 2$ they also showed that $S(a, a+1) = a^2$, and that $S(a, a+1)$ is not achieved, while for $a < b < a+1$, $S(a,b)$ is again achieved. For $n = 1$, $S(a, a+1) = (\frac{1+2a}{2})^2$ is also not achieved.
In this section we give a necessary and sufficient condition for improvement of (50) with $b = a+1$ and $n \geq 1$. Our results cover also the critical case when $a = \frac{n-2}{2}$ which is not allowed by the methods of [9].

Theorem 2.15 *Let W be a positive radial function on the ball B in $\mathbb{R}^n$ ($n \geq 1$) with radius R and centered at zero. Assume $a \leq \frac{n-2}{2}$. The following two statements are then equivalent:*

1. *W is a Bessel potential on $(0, R)$.*
2. *There exists $c > 0$ such that the following inequality holds for all $u \in C_0^\infty(B)$*

$$(\mathrm{H}_{a,cW}) \qquad \int_B |x|^{-2a} |\nabla u(x)|^2 dx \geq (\tfrac{n-2a-2}{2})^2 \int_B |x|^{-2a-2} u^2 dx + c \int_B |x|^{-2a} W(x) u^2 dx,$$

Moreover, $(\frac{n-2a-2}{2})^2$ is the best constant and $\beta(W;R) = \sup\{c; (H_{a,cW}) holds\}$, where $\beta(W;R)$ is the weight of the Bessel potential W on $(0,R)$.
On the other hand, there is no strictly positive $W \in C^1(0,\infty)$, such that the following inequality holds for all $u \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |x|^{-2a} |\nabla u(x)|^2 dx \geq (\frac{n-2a-2}{2})^2 \int_{\mathbb{R}^n} |x|^{-2a-2} u^2 dx + c \int_{\mathbb{R}^n} W(|x|) u^2 dx. \tag{55}$$

Proof: It suffices to use Theorems 2.1 and 2.9 with $V(r) = r^{-2a}$ to get that W is a Bessel function if and only if the pair $\left(r^{-2a}, W_{a,c}(r)\right)$ is a Bessel pair on $(0,R)$ for some $c > 0$, where

$$W_{a,c}(r) = (\frac{n-2a-2}{2})^2 r^{-2-2a} + c r^{-2a} W(r).$$

For the last part, assume that (55) holds for some W. Then it follows from Theorem 2.15 that for $V = cr^{2a} W(r)$ the equation $y''(r) + \frac{1}{r} y' + v(r) y = 0$ has a positive solution on $(0,\infty)$. From Lemma 2.7 we know that y is strictly decreasing on $(0, +\infty)$. Hence, $\frac{y''(r)}{y'(r)} \geq -\frac{1}{r}$ which yields $y'(r) \leq \frac{b}{r}$, for some $b > 0$. Thus $y(r) \to -\infty$ as $r \to +\infty$. This is a contradiction and the proof is complete. □

Remark 2.16 Theorem 2.15 characterizes the best constant only when Ω is a ball, while for general domain Ω, it just gives a lower and upper bounds for the best constant corresponding to a given Bessel potential W. It is indeed clear that

$$C_{B_R}(W) \leq C_\Omega(W) \leq C_{B_\rho}(W),$$

where B_R is the smallest ball containing Ω and B_ρ is the largest ball contained in it. If now W is a Bessel potential such that $\beta(W,R)$ is independent of R, then clearly $\beta(W,R)$ is also the best constant in inequality $(H_{a,cW})$ for any smooth bounded domain. This is clearly the case for the potentials $W_{k,\rho}$ and $\tilde{W}_{k,\rho}$ where $\beta(W,R) = \frac{1}{4}$ for all R, while for $W \equiv 1$ the best constant is still not known for general domains even for the simplest case $a = 0$.

Using the integral criteria for Bessel potentials, we can also deduce immediately the following.

Corollary 2.17 *Let Ω be a bounded smooth domain in R^n with $n \geq 1$, and let W be a non-negative function in $C^1(0, R =: \sup_{x\in\partial\Omega}|x|]$ and $a \leq \frac{n-2}{2}$.*

1. *If $\liminf_{r\to 0} \ln(r)\int_0^r sW(s)ds > -\infty$, then there exists $\alpha := \alpha(\Omega) > 0$ such that an improved Hardy inequality $(\mathrm{H}_{a,W_\alpha})$ holds for the scaled potential $W_\alpha(x) := \alpha^2 W(\alpha|x|)$.*
2. *If $\lim_{r\to 0} \ln(r)\int_0^r sW(s)ds = -\infty$, then there are no $\alpha, c > 0$, for which $(\mathrm{H}_{a,W_{\alpha,c}})$ holds with $W_{\alpha,c} = cW(\alpha|x|)$.*

By applying the above to various examples of Bessel potentials, we can now deduce several old and new inequalities. The first is an extension of a result established by Brezis and Vázquez [8] in the case where $a = 0$, and $b = 0$.

Corollary 2.18 *Let Ω be a bounded smooth domain in R^n with $n \geq 1$ and $a \leq \frac{n-2}{2}$. Then, for any $b < 2a+2$ there exists $c > 0$ such that for all $u \in C_0^\infty(\Omega)$*

$$\int_\Omega |x|^{-2a}|\nabla u|^2 dx \geq (\tfrac{n-2a-2}{2})^2 \int_\Omega |x|^{-2a-2}u^2 dx + c\int_\Omega |x|^{-b}u^2 dx. \tag{56}$$

Moreover, when Ω is a ball B of radius R the best constant c for which (56) holds is equal to the weight $\beta(r^{2a-b}; R)$ of the Bessel potential $W(r) = r^{2a-b}$ on $(0, R]$.
In particular,

$$\int_B |x|^{-2a}|\nabla u|^2 dx \geq (\tfrac{n-2a-2}{2})^2 \int_B |x|^{-2a-2}u^2 dx + \lambda_B \int_B |x|^{-2a}u^2 dx, \tag{57}$$

where the best constant λ_B is equal to $z_0\omega_n^{2/n}|\Omega|^{-2/n}$, where ω_n and $|\Omega|$ denote the volume of the unit ball and Ω respectively, and $z_0 = 2.4048...$ is the first zero of the Bessel function $J_0(z)$.

Proof: It suffices to apply Theorem 2.15 with the function $W(r) = r^{b+2a}$ which is a Bessel potential whenever $b > -2a-2$ since then $\liminf_{r\to 0}\ln(r)\int_0^r s^{2a+1}W(s)ds > -\infty$. In the case where $b = -2a$ and therefore $W \equiv 1$, we use the fact that $\beta(1; R) = \frac{z_0^2}{R^2}$ (established in the appendix) to deduce that the best constant is then equal to $z_0\omega_n^{2/n}|\Omega|^{-2/n}$. □

The following corollary is an extension of a recent result by Adimurthi et all [1] established in the case where $a = 0$, and of another result by Wang and Willem in [27] (Theorem 2) in the case $k = 1$. We also provide here the value of the best constant.

Corollary 2.19 *Let B be a bounded smooth domain in R^n with $n \geq 1$ and $a \leq \frac{n-2}{2}$. Then for every integer k, and $\rho = (\sup_{x\in\Omega}|x|)(e^{e^{e^{\cdot^{\cdot^{e((k-1)-times)}}}}})$, we have for any $u \in H_0^1(\Omega)$,*

$$\int_\Omega |x|^{-2a}|\nabla u|^2 dx \geq (\tfrac{n-2a-2}{2})^2 \int_\Omega \frac{u^2}{|x|^{2a+2}}dx + \frac{1}{4}\sum_{j=1}^k \int_\Omega \frac{|u|^2}{|x|^{2a+2}}\Big(\prod_{i=1}^j log^{(i)}\frac{\rho}{|x|}\Big)^{-2} dx. \tag{58}$$

Moreover, $\frac{1}{4}$ is the best constant which is not attained in $H_0^1(\Omega)$.

Proof: As seen in the appendix, $W_{k,\rho}(r) = \sum_{j=1}^k \frac{1}{r^2}\Big(\prod_{i=1}^j log^{(i)}\frac{\rho}{|x|}\Big)^{-2}dx$ is a Bessel potential on $(0, R)$ where $R = \sup_{x\in\Omega}|x|$, and $\beta(W_{k,\rho}; R) = \frac{1}{4}$. □

The very same reasoning leads to the following extension of a result established by Filippas and Tertikas [13] in the case where $a = 0$.

Corollary 2.20 *Let Ω be a bounded smooth domain in R^n with $n \geq 1$ and $a \leq \frac{n-2}{2}$. Then for every integer k, and any $D \geq \sup_{x\in\Omega}|x|$, we have for $u \in H_0^1(\Omega)$,*

$$\int_\Omega \frac{|\nabla u|^2}{|x|^{2a}}dx \geq (\tfrac{n-2a-2}{2})^2 \int_\Omega \frac{u^2}{|x|^{2a+2}}dx + \frac{1}{4}\sum_{i=1}^\infty \int_\Omega \frac{1}{|x|^{2a+2}} X_1^2(\tfrac{|x|}{D})X_2^2(\tfrac{|x|}{D})...X_i^2(\tfrac{|x|}{D})|u|^2 dx, \tag{59}$$

and $\frac{1}{4}$ is the best constant which is not attained in $H_0^1(\Omega)$.

The classical Hardy inequality is valid for dimensions $n \geq 3$. We now present optimal Hardy type inequalities for dimension two in bounded domains, as well as the corresponding best constants.

Theorem 2.21 *Let Ω be a smooth domain in R^2 and $0 \in \Omega$. Then we have the following inequalities.*

- *Let $D \geq \sup_{x\in\Omega}|x|$, then for all $u \in H_0^1(\Omega)$,*

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\sum_{i=1}^{\infty}\int_\Omega \frac{1}{|x|^2}X_1^2(\frac{|x|}{D})X_2^2(\frac{|x|}{D})...X_i^2(\frac{|x|}{D})|u|^2 dx \tag{60}$$

  and $\frac{1}{4}$ is the best constant.

- *Let $\rho = (\sup_{x\in\Omega}|x|)(e^{e^{e^{\cdot^{\cdot^{e((k-1)-times)}}}}})$, then for all $u \in H_0^1(\Omega)$*

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\sum_{j=1}^{k}\int_\Omega \frac{|u|^2}{|x|^2}\Big(\prod_{i=1}^{j} log^{(i)}\frac{\rho}{|x|}\Big)^{-2}dx, \tag{61}$$

  and $\frac{1}{4}$ is the best constant for all $k \geq 1$.

- *If $\alpha < 2$, then there exists $c > 0$ such that for all $u \in H_0^1(\Omega)$,*

$$\int_\Omega |\nabla u|^2 dx \geq c\int_\Omega \frac{u^2}{|x|^\alpha}\, dx, \tag{62}$$

  and the best constant is larger or equal to $\beta(r^\alpha; \sup\limits_{x\in\Omega}|x|)$.

An immediate application of Theorem 2.1 coupled with Hölder's inequality gives the following duality statement, which should be compared to inequalities dual to those of Sobolev's, recently obtained via the theory of mass transport [3, 10].

Corollary 2.22 *Suppose that Ω is a smooth bounded domain containing 0 in R^n $(n \geq 1)$ with $R := \sup_{x\in\Omega}|x|$. Then, for any $a \leq \frac{n-2}{2}$ and $0 < p \leq 2$, we have the following dual inequalities:*

$$\inf\left\{\int_\Omega |x|^{-2a}|\nabla u|^2 dx - (\frac{n-2a-2}{2})^2\int_\Omega |x|^{-2a-2}|u|^2 dx;\ u \in C_0^\infty(\Omega), ||u||_p = 1\right\}$$
$$\geq \sup\left\{\Big(\int_\Omega (\frac{|x|^{-2a}}{W(x)})^{\frac{p}{p-2}}\, dx\Big)^{\frac{2-p}{p}};\ W \in \mathcal{B}(0,R)\right\}.$$

3 General Hardy-Rellich inequalities

Let $0 \in \Omega \subset R^n$ be a smooth domain, and denote

$$C_{0,r}^k(\Omega) = \{v \in C_0^k(\Omega) : \text{v is radial and supp v } \subset \Omega\},$$

$$H_{0,r}^m(\Omega) = \{u \in H_0^m(\Omega) : \text{u is radial}\}.$$

We start by considering a general inequality for radial functions.

Theorem 3.1 *Let V and W be positive radial C^1-functions on a ball $B\backslash\{0\}$, where B is a ball with radius R in $\mathbb{R}^n$ $(n \geq 1)$ and centered at zero. Assume $\int_0^R \frac{1}{r^{n-1}V(r)}dr = \infty$ and $\lim_{r\to 0} r^\alpha V(r) = 0$ for some $\alpha < n-2$. Then the following statements are equivalent:*

1. *(V, W) is a Bessel pair on $(0, R)$.*
2. *There exists $c > 0$ such that the following inequality holds for all radial functions $u \in C_{0,r}^\infty(B)$*

$$(\mathrm{HR}_{V,cW}) \qquad \int_B V(x)|\Delta u|^2 dx \geq c\int_B W(x)|\nabla u|^2 dx + (n-1)\int_B (\frac{V(x)}{|x|^2} - \frac{V_r(|x|)}{|x|})|\nabla u|^2 dx.$$

Moreover, the best constant is given by

$$\beta(V,W;R)=\sup\big\{c;\ (\mathrm{HR}_{V,cW}) \text{ holds for radial functions}\big\}. \tag{63}$$

Proof: Assume $u\in C^\infty_{0,r}(B)$ and observe that

$$\int_B V(x)|\Delta u|^2dx=n\omega_n\{\int_0^R V(r)u_{rr}^2r^{n-1}dr+(n-1)^2\int_0^R V(r)\frac{u_r^2}{r^2}r^{n-1}dr+2(n-1)\int_0^R V(r)uu_rr^{n-2}dr\}.$$

Setting $\nu=u_r$, we then have

$$\int_B V(x)|\Delta u|^2dx=\int_B V(x)|\nabla\nu|^2dx+(n-1)\int_B(\frac{V(|x|)}{|x|^2}-\frac{V_r(|x|)}{|x|})|\nu|^2dx.$$

Thus, $(\mathrm{HR}_{V,W})$ for radial functions is equivalent to

$$\int_B V(x)|\nabla\nu|^2dx\geq\int_B W(x)\nu^2dx.$$

Letting $x(r)=\nu(x)$ where $|x|=r$, we then have

$$\int_0^R V(r)(x'(r))^2r^{n-1}dr\geq\int_0^R W(r)x^2(r)r^{n-1}dr. \tag{64}$$

It therefore follows from Theorem 2.1 that 1) and 2) are equivalent. □

By applying the above theorem to the Bessel pair

$$V(x)=|x|^{-2m}\quad\text{ and }\quad W_m(x)=V(x)\big[(\tfrac{n-2m-2}{2})^2|x|^{-2}+W(x)\big]$$

where W is a Bessel potential, and by using Theorem 2.9, we get the following result in the case of radial functions.

Corollary 3.2 *Suppose $n\geq1$ and $m<\frac{n-2}{2}$. Let $B_R\subset\mathbb{R}^n$ be a ball of radius $R>0$ and centered at zero. Let W be a Bessel potential on $(0,R)$. Then we have for all $u\in C^\infty_{0,r}(B_R)$*

$$\int_{B_R}\frac{|\Delta u|^2}{|x|^{2m}}\geq(\frac{n+2m}{2})^2\int_{B_R}\frac{|\nabla u|^2}{|x|^{2m+2}}dx+\beta(W;R)\int_{B_R}W(x)\frac{|\nabla u|^2}{|x|^{2m}}dx. \tag{65}$$

Moreover, $(\frac{n+2m}{2})^2$ and $\beta(W;R)$ are the best constants.

3.1 The non-radial case

The decomposition of a function into its spherical harmonics will be one of our tools to prove the corresponding result in the non-radial case. This idea has also been used in [24]. Any function $u\in C_0^\infty(\Omega)$ could be extended by zero outside Ω, and could therefore be considered as a function in $C_0^\infty(R^n)$. By decomposing u into spherical harmonics we get

$$u=\Sigma_{k=0}^\infty u_k \text{ where } u_k=f_k(|x|)\varphi_k(x)$$

and $(\varphi_k(x))_k$ are the orthonormal eigenfunctions of the Laplace-Beltrami operator with corresponding eigenvalues $c_k=k(n+k-2)$, $k\geq0$. The functions f_k belong to $C_0^\infty(\Omega)$ and satisfy $f_k(r)=O(r^k)$ and $f'(r)=O(r^{k-1})$ as $r\to0$. In particular,

$$\varphi_0=1 \text{ and } f_0=\tfrac{1}{n\omega_nr^{n-1}}\textstyle\int_{\partial B_r}uds=\tfrac{1}{n\omega_n}\int_{|x|=1}u(rx)ds. \tag{66}$$

We also have for any $k\geq0$, and any continuous real valued functions v and w on $(0,\infty)$,

$$\int_{R^n}V(|x|)|\Delta u_k|^2dx=\int_{R^n}V(|x|)\big(\Delta f_k(|x|)-c_k\frac{f_k(|x|)}{|x|^2}\big)^2dx, \tag{67}$$

and

$$\int_{R^n}W(|x|)|\nabla u_k|^2dx=\int_{R^n}W(|x|)|\nabla f_k|^2dx+c_k\int_{R^n}W(|x|)|x|^{-2}f_k^2dx. \tag{68}$$

Theorem 3.3 *Let V and W be positive radial C^1-functions on a ball $B\backslash\{0\}$, where B is a ball with radius R in $\mathbb{R}^n$ $(n \geq 1)$ and centered at zero. Assume $\int_0^R \frac{1}{r^{n-1}V(r)}dr = \infty$ and $\lim_{r\to 0} r^\alpha V(r) = 0$ for some $\alpha < (n-2)$. If*

$$W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r) \geq 0 \quad for \quad 0 \leq r \leq R, \tag{69}$$

then the following statements are equivalent.

1. *(V, W) is a Bessel pair with $\beta(V, W; R) \geq 1$.*
2. *The following inequality holds for all $u \in C_0^\infty(B)$,*

$$(\mathrm{HR}_{V,W}) \qquad \int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B \left(\frac{V(x)}{|x|^2} - \frac{V_r(|x|)}{|x|}\right)|\nabla u|^2 dx.$$

Moreover, if $\beta(V, W; R) \geq 1$, then the best constant is given by

$$\beta(V, W; R) = \sup\big\{c;\ (\mathrm{HR}_{V,cW})\ holds\big\}. \tag{70}$$

Proof: That 2) implies 1) follows from Theorem 3.1 and does not require condition (69). To prove that 1) implies 2) assume that the equation $(B_{V,W})$ has a positive solution on $(0, R]$. We prove that the inequality $(HR_{V,W})$ holds for all $u \in C_0^\infty(B)$ by frequently using that

$$\int_0^R V(r)|x'(r)|^2 r^{n-1} dr \geq \int_0^R W(r) x^2(r) r^{n-1} dr \text{ for all } x \in C^1(0, R]. \tag{71}$$

Indeed, for all $n \geq 1$ and $k \geq 0$ we have

$$\begin{aligned}
\frac{1}{nw_n}\int_{R^n} V(x)|\Delta u_k|^2 dx &= \frac{1}{nw_n}\int_{R^n} V(x)\big(\Delta f_k(|x|) - c_k\frac{f_k(|x|)}{|x|^2}\big)^2 dx \\
&= \int_0^R V(r)\big(f_k''(r) + \frac{n-1}{r}f_k'(r) - c_k\frac{f_k(r)}{r^2}\big)^2 r^{n-1} dr \\
&= \int_0^R V(r)(f_k''(r))^2 r^{n-1} dr + (n-1)^2\int_0^R V(r)(f_k'(r))^2 r^{n-3} dr \\
&\quad + c_k^2\int_0^R V(r) f_k^2(r) r^{n-5} + 2(n-1)\int_0^R V(r) f_k''(r) f_k'(r) r^{n-2} \\
&\quad - 2c_k\int_0^R V(r) f_k''(r) f_k(r) r^{n-3} dr - 2c_k(n-1)\int_0^R V(r) f_k'(r) f_k(r) r^{n-4} dr.
\end{aligned}$$

Integrate by parts and use (66) for $k = 0$ to get

$$\begin{aligned}
\frac{1}{n\omega_n}\int_{R^n} V(x)|\Delta u_k|^2 dx &= \int_0^R V(r)(f_k''(r))^2 r^{n-1} dr + (n-1+2c_k)\int_0^R V(r)(f_k'(r))^2 r^{n-3} dr \\
&+ (2c_k(n-4) + c_k^2)\int_0^R V(r) r^{n-5} f_k^2(r) dr - (n-1)\int_0^R V_r(r) r^{n-2} (f_k')^2(r) dr \\
&- c_k(n-5)\int_0^R V_r(r) f_k^2(r) r^{n-4} dr - c_k\int_0^R V_{rr}(r) f_k^2(r) r^{n-3} dr.
\end{aligned} \tag{72}$$

Now define $g_k(r) = \frac{f_k(r)}{r}$ and note that $g_k(r) = O(r^{k-1})$ for all $k \geq 1$. We have

$$\begin{aligned}
\int_0^R V(r)(f_k'(r))^2 r^{n-3} &= \int_0^R V(r)(g_k'(r))^2 r^{n-1} dr + \int_0^R 2V(r) g_k(r) g_k'(r) r^{n-2} dr + \int_0^R V(r) g_k^2(r) r^{n-3} dr \\
&= \int_0^R V(r)(g_k'(r))^2 r^{n-1} dr - (n-3)\int_0^R V(r) g_k^2(r) r^{n-3} dr - \int_0^R V_r(r) g_k^2(r) r^{n-2} dr
\end{aligned}$$

Thus,

$$\int_0^R V(r)(f_k'(r))^2 r^{n-3} \geq \int_0^R W(r) f_k^2(r) r^{n-3} dr - (n-3)\int_0^R V(r) f_k^2(r) r^{n-5} dr - \int_0^R V_r(r) f_k^2(r) r^{n-4} dr. \quad (73)$$

Substituting $2c_k \int_0^R V(r)(f_k'(r))^2 r^{n-3}$ in (72) by its lower estimate in the last inequality (73), we get

$$\begin{aligned}
\frac{1}{n\omega_n}\int_{R^n} V(x)|\Delta u_k|^2 dx \quad &\geq \quad \int_0^R W(r)(f_k'(r))^2 r^{n-1} dr + \int_0^R W(r)(f_k(r))^2 r^{n-3} dr \\
&+ \quad (n-1)\int_0^R V(r)(f_k'(r))^2 r^{n-3} dr + c_k(n-1)\int_0^R V(r)(f_k(r))^2 r^{n-5} dr \\
&- \quad (n-1)\int_0^R V_r(r) r^{n-2}(f_k')^2(r) dr - c_k(n-1)\int_0^R V_r(r) r^{n-4}(f_k)^2(r) dr \\
&+ \quad c_k(c_k-(n-1))\int_0^R V(r) r^{n-5} f_k^2(r) dr \\
&+ \quad c_k \int_0^R (W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r)) f_k^2(r) r^{n-3} dr.
\end{aligned}$$

The proof is now complete since the last term is non-negative by condition (69). Note also that because of this condition, the formula for the best constant requires that $\beta(V, W; R) \geq 1$, since if W satisfies (69) then cW satisfies it for any $c \geq 1$. $\square$

Remark 3.4 In order to apply the above theorem to the Bessel pair

$$V(x) = |x|^{-2m} \quad \text{ and } \quad W_{m,c}(x) = V(x)\big[(\tfrac{n-2m-2}{2})^2|x|^{-2} + cW(x)\big]$$

where W is a Bessel potential, we see that even in the simplest case $V \equiv 1$ and $W_{m,c}(x) = (\frac{n-2}{2})^2|x|^{-2} + W(x)$, condition (69) reduces to $(\frac{n-2}{2})^2|x|^{-2} + W(x) \geq 2|x|^{-2}$, which is then guaranteed only if $n \geq 5$.
More generally, if $V(x) = |x|^{-2m}$, then in order to satisfy (69) we need to have

$$\frac{-(n+4) - 2\sqrt{n^2-n+1}}{6} \leq m \leq \frac{-(n+4) + 2\sqrt{n^2-n+1}}{6}, \quad (74)$$

and in this case, we have for $m < \frac{n-2}{2}$ and any Bessel potential W on B_R, that for all $u \in C_0^\infty(B_R)$

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \geq (\frac{n+2m}{2})^2 \int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx + \beta(W; R)\int_{B_R} W(x)\frac{|\nabla u|^2}{|x|^{2m}} dx. \quad (75)$$

Moreover, $(\frac{n+2m}{2})^2$ and $\beta(W; R)$ are the best constant.
Therefore, inequality (75) in the case where $m = 0$ and $n \geq 5$, already includes Theorem 1.5 in [24] as a special case. It also extends Theorem 1.8 in [24] where it is established under the condition

$$0 \leq m \leq \frac{-(n+4) + 2\sqrt{n^2-n+1}}{6} \quad (76)$$

which is more restrictive than (74). We shall see however that this inequality remains true without condition (74), but with a constant that is sometimes different from $(\frac{n+2m}{2})^2$ in the cases where (74) is not valid. For example, if $m = 0$, then the best constant is 3 in dimension 4 and $\frac{25}{36}$ in dimension 3.

We shall now give a few immediate applications of the above in the case where $m = 0$ and $n \geq 5$. Actually the results are true in lower dimensions, and will be stated as such, but the proofs for $n < 5$ will require additional work and will be postponed to the next section.

Theorem 3.5 *Assume W is a Bessel potential on $B_R \subset R^n$ with $n \geq 3$, then for all $u \in C_0^\infty(B_R)$ we have*

$$\int_{B_R} |\Delta u|^2 dx \geq C(n) \int_{B_R} \frac{|\nabla u|^2}{|x|^2} dx + \beta(W;R) \int_{B_R} W(x)|\nabla u|^2 dx, \tag{77}$$

where $C(3) = \frac{25}{36}$, $C(4) = 3$ and $C(n) = \frac{n^2}{4}$ for all $n \geq 5$. Moreover, $C(n)$ and $\beta(W;R)$ are the best constants.
In particular, the following holds for any smooth bounded domain Ω in R^n with $R = \sup_{x\in\Omega}|x|$, and any $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

- *For any $\alpha < 2$,*

$$\int_{\Omega} |\Delta u|^2 dx \geq C(n) \int_{\Omega} \frac{|\nabla u|^2}{|x|^2} dx + \beta(|x|^\alpha;R) \int_{\Omega} \frac{|\nabla u|^2}{|x|^\alpha} dx, \tag{78}$$

  and for $\alpha = 0$,

$$\textstyle\int_{\Omega} |\Delta u|^2 dx \geq C(n) \int_{\Omega} \frac{|\nabla u|^2}{|x|^2} dx + \frac{z_0^2}{R^2} \int_{\Omega} |\nabla u|^2 dx, \tag{79}$$

  the constants being optimal when Ω is a ball.

- *For any $k \geq 1$, and $\rho = R(e^{e^{e^{\cdot^{\cdot^{e(k-times)}}}}})$, we have*

$$\int_{\Omega} |\Delta u(x)|^2 dx \geq C(n) \int_{\Omega} \frac{|\nabla u|^2}{|x|^2} dx + \frac{1}{4}\sum_{j=1}^{k} \int_{\Omega} \frac{|\nabla u|^2}{|x|^2} \Big(\prod_{i=1}^{j} log^{(i)} \frac{\rho}{|x|}\Big)^{-2} dx, \tag{80}$$

- *For $D \geq R$, and X_i is defined as (106) we have*

$$\int_{\Omega} |\Delta u(x)|^2 dx \geq C(n) \int_{\Omega} \frac{|\nabla u|^2}{|x|^2} dx + \frac{1}{4}\sum_{i=1}^{\infty} \int_{\Omega} \frac{|\nabla u|}{|x|^2} X_1^2\Big(\frac{|x|}{D}\Big) X_2^2\Big(\frac{|x|}{D}\Big) \ldots X_i^2\Big(\frac{|x|}{D}\Big) dx, \tag{81}$$

  Moreover, all constants appearing in the above two inequality are optimal.

Theorem 3.6 *Let $W(x) = W(|x|)$ be radial Bessel potential on a ball B of radius R in R^n with $n \geq 4$, and such that $\frac{W_r(r)}{W(r)} = \frac{\lambda}{r} + f(r)$, where $f(r) \geq 0$ and $\lim_{r\to 0} rf(r) = 0$. If $\lambda < n-2$, then the following Hardy-Rellich inequality holds:*

$$\int_B |\Delta u|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_B \frac{u^2}{|x|^4} dx + \Big(\frac{n^2}{4} + \frac{(n-\lambda-2)^2}{4}\Big)\beta(W;R) \int_B \frac{W(x)}{|x|^2} u^2 dx, \tag{82}$$

Proof: Use first Theorem 3.5 with the Bessel potential W, then Theorem 2.15 with the Bessel pair $(|x|^{-2}, |x|^{-2}(\frac{(n-4)^2}{4}|x|^{-2} + W)$, then Theorem 2.9 with the Bessel pair $(W, \frac{(n-\lambda-2)^2}{4})|x|^{-2}W)$ to obtain

$$\begin{aligned}
\int_B |\Delta u|^2 dx &\geq C(n) \int_B \frac{|\nabla u|^2}{|x|^2} dx + \beta(W,R) \int_B W(x)|\nabla u|^2 dx \\
&\geq C(n)\frac{(n-4)^2}{4} \int_B \frac{u^2}{|x|^4} dx + C(n)\beta(W,R) \int_B \frac{W(x)}{|x|^2} u^2 + \beta(W,R) \int W(x)|\nabla u|^2 dx \\
&\geq C(n)\frac{(n-4)^2}{4} \int_B \frac{u^2}{|x|^4} dx + \Big(C(n) + \frac{(n-\lambda-2)^2}{4}\Big)\beta(W,R) \int_B \frac{W(x)}{|x|^2} u^2 dx.
\end{aligned}$$

Recall that $C(n) = \frac{n^2}{4}$ for $n \geq 5$, giving the claimed result in these dimensions. This is however not the case when $n = 4$, and therefore another proof will be given in the next section to cover these cases.
The following is immediate from Theorem 3.5 and from the fact that $\lambda = 2$ for the Bessel potential under consideration.

Corollary 3.7 *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 4$ and $R = \sup_{x\in\Omega}|x|$. Then the following holds for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$*

1. *If* $\rho = R(e^{e^{e^{\cdot^{\cdot^{e(k-times)}}}}})$ *and* $log^{(i)}(.)$ *is defined as (105), then*

$$\int_\Omega |\Delta u(x)|^2 dx \geq \frac{n^2(n-4)^2}{16}\int_\Omega \frac{u^2}{|x|^4}dx + (1+\frac{n(n-4)}{8})\sum_{j=1}^{k}\int_\Omega \frac{u^2}{|x|^4}(\prod_{i=1}^{j} log^{(i)}\frac{\rho}{|x|})^{-2}dx. \tag{83}$$

2. *If* $D \geq R$ *and* X_i *is defined as (106), then*

$$\int_\Omega |\Delta u(x)|^2 dx \geq \frac{n^2(n-4)^2}{16}\int_\Omega \frac{u^2}{|x|^4}dx + (1+\frac{n(n-4)}{8})\sum_{i=1}^{\infty}\int_\Omega \frac{u^2}{|x|^4}X_1^2(\frac{|x|}{D})X_2^2(\frac{|x|}{D})...X_i^2(\frac{|x|}{D})dx. \tag{84}$$

Theorem 3.8 *Let* $W_1(x)$ *and* $W_2(x)$ *be two radial Bessel potentials on a ball* B *of radius* R *in* R^n *with* $n \geq 4$. *If* $a < 1$, *then there exists* $c(a, R) > 0$ *such that for all* $u \in H^2(B) \cap H_0^1(B)$

$$\begin{aligned}\int_B |\Delta u|^2 dx &\geq \frac{n^2(n-4)^2}{16}\int_B \frac{u^2}{|x|^4}dx + \frac{n^2}{4}\beta(W_1;R)\int_B W_1(x)\frac{u^2}{|x|^2}dx \\ &\quad + c(\frac{n-2a-2}{2})^2\int_B \frac{u^2}{|x|^{2a+2}}dx + c\beta(W_2;R)\int_B W_2(x)\frac{u^2}{|x|^{2a}}dx,\end{aligned}$$

Proof: Here again we shall give a proof when $n \geq 5$. The case $n = 4$ will be handled in the next section. We again first use Theorem 3.5 (for $n \geq 5$) with the Bessel potential $|x|^{-2a}$ where $a < 1$, then Theorem 2.15 with the Bessel pair $(|x|^{-2}, |x|^{-2}(\frac{(n-4)^2}{4}|x|^{-2} + W))$, then again Theorem 2.15 with the Bessel pair $(|x|^{-2a}, |x|^{-2a}((\frac{n-2a-2}{2})^2|x|^{-2} + W)$ to obtain

$$\begin{aligned}\int_B |\Delta u|^2 dx &\geq \frac{n^2}{4}\int_B \frac{|\nabla u|^2}{|x|^2}dx + \beta(|x|^{-2a};R)\int_B \frac{|\nabla u|^2}{|x|^{-2a}}dx \\ &\geq \frac{n^2(n-4)^2}{16}\int_B \frac{u^2}{|x|^4}dx + \frac{n^2}{4}\beta(W_1;R)\int_B W_1(x)\frac{u^2}{|x|^2}dx + \beta(|x|^{-2a};R)\int_B \frac{|\nabla u|^2}{|x|^{-2a}}dx \\ &\geq \frac{n^2(n-4)^2}{16}\int_B \frac{u^2}{|x|^4}dx + \frac{n^2}{4}\beta(W_1;R)\int_B W_1(x)\frac{u^2}{|x|^2}dx \\ &\quad + \beta(|x|^{-2a};R)(\frac{n-2a-2}{2})^2\int_B \frac{u^2}{|x|^{2a+2}}dx + \beta(|x|^{-2a};R)\beta(W_2;R)\int_B W_2(x)\frac{u^2}{|x|^{2a}}dx.\end{aligned}$$

The following theorem will be established in full generality (i.e with $V(r) = r^{-m}$) in the next section.

Theorem 3.9 *Let* $W(x) = W(|x|)$ *be a radial Bessel potential on a smooth bounded domain* Ω *in* $\mathbb{R}^n$, $n \geq 4$. *Then,*

$$\int_\Omega |\Delta u(x)|^2 dx - \frac{n^2(n-4)^2}{16}\int_\Omega \frac{u^2}{|x|^4}dx - \frac{n^2}{4}\int_\Omega W(x)u^2 dx \geq \frac{z_0^2}{R^2}||u||^2_{W_0^{1,2}(\Omega)} \quad u \in H_0^2(\Omega).$$

3.2 The case of power potentials $|x|^m$

The general Theorem 3.3 allowed us to deduce inequality (85) below for a restricted interval of powers m. We shall now prove that the same holds for all $m < \frac{n-2}{2}$. The following theorem improves considerably Theorem 1.7, Theorem 1.8, and Theorem 6.4 in [24].

Theorem 3.10 *Suppose* $n \geq 1$ *and* $m < \frac{n-2}{2}$, *and let* W *be a Bessel potential on a ball* $B_R \subset R^n$ *of radius* R. *Then for all* $u \in C_0^\infty(B_R)$

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \geq a_{n,m}\int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}}dx + \beta(W;R)\int_{B_R} W(x)\frac{|\nabla u|^2}{|x|^{2m}}dx, \tag{85}$$

where

$$a_{n,m} = \inf\left\{\frac{\int_{B_R}\frac{|\Delta u|^2}{|x|^{2m}}dx}{\int_{B_R}\frac{|\nabla u|^2}{|x|^{2m+2}}dx};\ u \in C_0^\infty(B_R)\setminus\{0\}\right\}.$$

Moreover, $\beta(W;R)$ *and* $a_{m,n}$ *are the best constants to be computed in the appendix.*

Proof: Assuming the inequality

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \geq a_{n,m} \int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx,$$

holds for all $u \in C_0^\infty(B_R)$, we shall prove that it can be improved by any Bessel potential W. We will use the following inequality frequently in the proof which follows directly from Theorem 2.15 with n=1.

$$\int_0^R r^\alpha (f'(r))^2 dr \geq (\frac{\alpha-1}{2})^2 \int_0^R r^{\alpha-2} f^2(r) dr + \beta(W;R) \int_0^R r^\alpha W(r) f^2(r) dr, \quad \alpha \geq 1, \tag{86}$$

for all $f \in C^\infty(0,R)$, where both $(\frac{\alpha-1}{2})^2$ and $\beta(W;R)$ are best constants.
Decompose $u \in C_0^\infty(B_R)$ into its spherical harmonics $\Sigma_{k=0}^\infty u_k$, where $u_k = f_k(|x|)\varphi_k(x)$. We evaluate $I_k = \frac{1}{nw_n} \int_{R^n} \frac{|\Delta u_k|^2}{|x|^{2m}} dx$ in the following way

$$\begin{aligned}
I_k &= \int_0^R r^{n-2m-1} (f_k''(r))^2 dr + [(n-1)(2m+1) + 2c_k] \int_0^R r^{n-2m-3} (f_k')^2 dr \\
&\quad + c_k[c_k + (n-2m-4)(2m+2)] \int_0^R r^{n-2m-5} (f_k(r))^2 dr \\
&\geq \beta(W) \int_0^R r^{n-2m-1} W(x) (f_k')^2 dr + [(\frac{n+2m}{2})^2 + 2c_k] \int_0^R r^{n-2m-3} (f_k')^2 dr \\
&\quad + c_k[c_k + (n-2m-4)(2m+2)] \int_0^R r^{n-2m-5} (f_k(r))^2 dr \\
&\geq \beta(W) \int_0^R r^{n-2m-1} W(x) (f_k')^2 dr + a_{n,m} \int_0^R r^{n-2m-3} (f_k')^2 dr \\
&\quad + \beta(W)[(\frac{n+2m}{2})^2 + 2c_k - a_{n,m}] \int_0^R r^{n-2m-3} W(x) (f_k)^2 dr \\
&\quad + \big((\frac{n-2m-4}{2})^2 [(\frac{n+2m}{2})^2 + 2c_k - a_{n,m}] + c_k[c_k + (n-2m-4)(2m+2)]\big) \int_0^R r^{n-2m-5} (f_k(r))^2 dr.
\end{aligned}$$

Now by (115) we have

$$((\frac{n-2m-4}{2})^2 [(\frac{n+2m}{2})^2 + 2c_k - a_{n,m}] + c_k[c_k + (n-2m-4)(2m+2)] \geq c_k a_{n,m},$$

for all $k \geq 0$. Hence, we have

$$\begin{aligned}
I_k &\geq a_{n,m} \int_0^R r^{n-2m-3} (f_k')^2 dr + a_{n,m} c_k \int_0^R r^{n-2m-5} (f_k(r))^2 dr \\
&\quad + \beta(W) \int_0^R r^{n-2m-1} W(x) (f_k')^2 dr + \beta(W)[(\frac{n+2m}{2})^2 + 2c_k - a_{n,m}] \int_0^R r^{n-2m-3} W(x) (f_k)^2 dr \\
&\geq a_{n,m} \int_0^R r^{n-2m-3} (f_k')^2 dr + a_{n,m} c_k \int_0^R r^{n-2m-5} (f_k(r))^2 dr \\
&\quad + \beta(W) \int_0^R r^{n-2m-1} W(x) (f_k')^2 dr + \beta(W) c_k \int_0^R r^{n-2m-3} W(x) (f_k)^2 dr \\
&= a_{n,m} \int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx + \beta(W) \int_{B_R} W(x) \frac{|\nabla u|^2}{|x|^{2m}} dx.
\end{aligned}$$

Moreover, it is easy to see from Theorem 2.1 and the above calculation that $\beta(W;R)$ is the best constant.

Theorem 3.11 *Let Ω be a smooth domain in R^n with $n \geq 1$ and let $V \in C^2(0, R =: \sup_{x\in\Omega}|x|)$ be a non-negative function that satisfies the following conditions:*

$$V_r(r) \leq 0 \quad \text{and} \quad \int_0^R \tfrac{1}{r^{n-3}V(r)}dr = -\int_0^R \tfrac{1}{r^{n-4}V_r(r)}dr = +\infty. \tag{87}$$

There exists $\lambda_1, \lambda_2 \in R$ such that

$$\tfrac{rV_r(r)}{V(r)} + \lambda_1 \geq 0 \text{ on } (0,R) \text{ and } \lim_{r\to 0} \tfrac{rV_r(r)}{V(r)} + \lambda_1 = 0, \tag{88}$$

$$\tfrac{rV_{rr}(r)}{V_r(r)} + \lambda_2 \geq 0 \text{ on } (0,R) \text{ and } \lim_{r\to 0} \tfrac{rV_{rr}(r)}{V_r(r)} + \lambda_2 = 0, \tag{89}$$

and

$$\left(\tfrac{1}{2}(n-\lambda_1-2)^2 + 3(n-3)\right)V(r) - (n-5)rV_r(r) - r^2V_{rr}(r) \geq 0 \text{ for all } r \in (0,R). \tag{90}$$

Then the following inequality holds:

$$\begin{aligned}\int_\Omega V(|x|)|\Delta u|^2dx \quad &\geq \quad (\frac{(n-\lambda_1-2)^2}{4} + (n-1))\frac{(n-\lambda_1-4)^2}{4}\int_\Omega \frac{V(|x|)}{|x|^4}u^2dx \\ &\quad - \frac{(n-1)(n-\lambda_2-2)^2}{4}\int_\Omega \frac{V_r(|x|)}{|x|^3}u^2dx.\end{aligned} \tag{91}$$

Proof: We have by Theorem 2.9 and condition (90),

$$\begin{aligned}\frac{1}{n\omega_n}\int_{R^n} V(x)|\Delta u_k|^2dx \quad &= \quad \int_0^R V(r)(f_k''(r))^2r^{n-1}dr + (n-1+2c_k)\int_0^R V(r)(f_k'(r))^2r^{n-3}dr \\ &+ \quad (2c_k(n-4)+c_k^2)\int_0^R V(r)r^{n-5}f_k^2(r)dr - (n-1)\int_0^R V_r(r)r^{n-2}(f_k')^2(r)dr \\ &- \quad c_k(n-5)\int_0^R V_r(r)f_k^2(r)r^{n-4}dr - c_k\int_0^R V_{rr}(r)f_k^2(r)r^{n-3}dr \\ &\geq \quad \int_0^R V(r)(f_k''(r))^2r^{n-1}dr + (n-1)\int_0^R V(r)(f_k'(r))^2r^{n-3}dr \\ &- \quad (n-1)\int_0^R V_r(r)r^{n-2}(f_k')^2(r)dr \\ &+ \quad c_k\int_0^R \left(\left(\frac{1}{2}(n-\lambda_1-2)^2+3(n-3)\right)V(r) - (n-5)rV_r(r) - r^2V_{rr}(r)\right)f_k^2(r)r^{n-5}dr\end{aligned}$$

The rest of the proof follows from the above inequality combined with Theorem 2.9. □

Remark 3.12 Let $V(r) = r^{-2m}$ with $m \leq \frac{n-4}{2}$. Then in order to satisfy condition (90) we must have $-1 - \frac{\sqrt{1+(n-1)^2}}{2} \leq m \leq \frac{n-4}{2}$. Under this assumption the inequality (91) gives the following weighted second order Rellich inequality:

$$\int_B \frac{|\Delta u|^2}{|x|^{2m}}dx \geq (\frac{(n+2m)(n-4-2m)}{4})^2\int_B \frac{u^2}{|x|^{2m+4}}dx.$$

In the following theorem we will show that the constant appearing in the above inequality is optimal. Moreover, we will see that if $m < -1 - \frac{\sqrt{1+(n-1)^2}}{2}$, then the best constant is strictly less than $(\frac{(n+2m)(n-4-2m)}{4})^2$. This shows that inequality (91) is actually sharp.

Theorem 3.13 *Let $m \leq \frac{n-4}{2}$ and define*

$$\beta_{n,m} = \inf_{u\in C_0^\infty(B)\setminus\{0\}} \frac{\int_B \frac{|\Delta u|^2}{|x|^{2m}}dx}{\int_B \frac{u^2}{|x|^{2m+4}}dx}. \tag{92}$$

Then

$$\beta_{n,m} = (\frac{(n+2m)(n-4-2m)}{4})^2 + \min_{k=0,1,2,\dots}\{k(n+k-2)[k(n+k-2) + \frac{(n+2m)(n-2m-4)}{2}]\}.$$

Consequently the values of $\beta_{n,m}$ are as follows.

1. *If* $-1-\frac{\sqrt{1+(n-1)^2}}{2} \le m \le \frac{n-4}{2}$, *then*

$$\beta_{n,m} = (\frac{(n+2m)(n-4-2m)}{4})^2.$$

2. *If* $\frac{n}{2}-3 \le m \le -1-\frac{\sqrt{1+(n-1)^2}}{2}$, *then*

$$\beta_{n,m} = (\frac{(n+2m)(n-4-2m)}{4})^2 + (n-1)[(n-1) + \frac{(n+2m)(n-2m-4)}{2}].$$

3. *If* $k := \frac{n-2m-4}{2} \in N$, *then*

$$\beta_{n,m} = (\frac{(n+2m)(n-4-2m)}{4})^2 + k(n+k-2)[k(n+k-2) + \frac{(n+2m)(n-2m-4)}{2}].$$

4. *If* $k < \frac{n-2m-4}{2} < k+1$ *for some* $k \in N$, *then*

$$\beta_{n,m} = \frac{(n+2m)^2(n-2m-4)^2}{16} + a(m,n,k)$$

where

$$a(m,n,k) = \min\left\{k(n+k-2)[k(n+k-2) + \frac{(n+2m)(n-2m-4)}{2}], (k+1)(n+k-1)[(k+1)(n+k-1) + \frac{(n+2m)(n-2m-4)}{2}]\right\}.$$

Proof: Decompose $u \in C_0^\infty(B_R)$ into spherical harmonics $\Sigma_{k=0}^\infty u_k$, where $u_k = f_k(|x|)\varphi_k(x)$. we have

$$\begin{aligned}
\frac{1}{n\omega_n}\int_{R^n} \frac{|\Delta u_k|^2}{|x|^{2m}}dx &= \int_0^R r^{n-2m-1}(f_k''(r))^2 dr + [(n-1)(2m+1)+2c_k]\int_0^R r^{n-2m-3}(f_k')^2 dr \\
&+ c_k[c_k + (n-2m-4)(2m+2)]\int_0^R r^{n-2m-5}(f_k(r))^2 dr \\
&\ge ((\frac{(n+2m)(n-4-2m)}{4})^2 \\
&+ c_k[c_k + \frac{(n+2m)(n-2m-4)}{2}])\int_0^R r^{n-2m-5}(f_k(r))^2 dr,
\end{aligned}$$

by Hardy inequality. Hence,

$$\beta_{n,m} \ge B(n,m,k) := (\frac{(n+2m)(n-4-2m)}{4})^2 + \min_{k=0,1,2,\ldots}\{k(n+k-2)[k(n+k-2) + \frac{(n+2m)(n-2m-4)}{2}]\}.$$

To prove that $\beta_{n,m}$ is the best constant, let k be such that

$$\beta_{n,m} = \frac{(n+2m)(n-4-2m)}{4})^2 + k(n+k-2)[k(n+k-2) + \frac{(n+2m)(n-2m-4)}{2}]. \tag{93}$$

Set

$$u = |x|^{-\frac{n-4}{2}+m+\epsilon}\varphi_k(x)\varphi(|x|),$$

where $\varphi_k(x)$ is an eigenfunction corresponding to the eigenvalue c_k and $\varphi(r)$ is a smooth cutoff function, such that $0 \le \varphi \le 1$, with $\varphi \equiv 1$ in $[0, \frac{1}{2}]$. We have

$$\frac{\int_{B_R}\frac{|\Delta u|^2}{|x|^{2m}}dx}{\int_{B_R}\frac{u^2}{|x|^{2m+4}}dx} = (-\frac{(n+2m)(n-4-2m)}{4} - c_k + \epsilon(2+2m+\epsilon))^2 + O(1).$$

Let now $\epsilon \to 0$ to obtain the result. Thus the inequality

$$\int_{B_R}\frac{|\Delta u|^2}{|x|^{2m}} \ge \beta_{n,m}\int_{B_R}\frac{u^2}{|x|^{2m+4}}dx,$$

holds for all $u \in C_0^\infty(B_R)$.
To calculate explicit values of $\beta_{n,m}$ we need to find the minimum point of the function

$$f(x) = x(x + \frac{(n+2m)(n-2m-4)}{2}), \quad x \geq 0.$$

Observe that

$$f'(-\frac{(n+2m)(n-2m-4)}{4}) = 0.$$

To find minimizer $k \in N$ we should solve the equation

$$k^2 + (n-2)k + \frac{(n+2m)(n-2m-4)}{4} = 0.$$

The roots of the above equation are $x_1 = \frac{n+2m}{2}$ and $x_2 = \frac{n-2m-4}{2}$. 1) follows from Theorem 3.11. It is easy to see that if $m \leq -1 - \frac{\sqrt{1+(n-1)^2}}{2}$, then $x_1 < 0$. Hence, for $m \leq -1 - \frac{\sqrt{1+(n-1)^2}}{2}$ the minimum of the function f is attained in x_2. Note that if $m \leq -1 - \frac{\sqrt{1+(n-1)^2}}{2}$, then $B(n, m1) \leq B(n, m, 0)$. Therefore claims 2), 3), and 4) follow. □

The following theorem extends Theorem 1.6 of [24] in many ways. First, we do not assume that $n \geq 5$ or $m \geq 0$, as was assumed there. Moreover, inequality (94) below includes inequalities (1.17) and (1.22) of [24] as special cases.

Theorem 3.14 *Let $m \leq \frac{n-4}{2}$ and let $W(x)$ be a Bessel potential on a ball B of radius R in R^n with radius R. Assume $\frac{W(r)}{W_r(r)} = -\frac{\lambda}{r} + f(r)$, where $f(r) \geq 0$ and $\lim_{r \to 0} r f(r) = 0$. Then the following inequality holds for all $u \in C_0^\infty(B)$*

$$\begin{aligned} \int_B \frac{|\Delta u|^2}{|x|^{2m}} dx \quad &\geq \quad \beta_{n,m} \int_B \frac{u^2}{|x|^{2m+4}} dx \\ &+\beta(W;R)(\frac{(n+2m)^2}{4} + \frac{(n-2m-\lambda-2)^2}{4}) \int_B \frac{W(x)}{|x|^{2m+2}} u^2 dx. \end{aligned} \tag{94}$$

Proof: Again we will frequently use inequality (86) in the proof. Decomposing $u \in C_0^\infty(B_R)$ into spherical harmonics $\Sigma_{k=0}^\infty u_k$, where $u_k = f_k(|x|)\varphi_k(x)$, we can write

$$\begin{aligned} \frac{1}{n\omega_n} \int_{R^n} \frac{|\Delta u_k|^2}{|x|^{2m}} dx \quad &= \quad \int_0^R r^{n-2m-1} (f_k''(r))^2 dr + [(n-1)(2m+1) + 2c_k] \int_0^R r^{n-2m-3} (f_k')^2 dr \\ &+c_k[c_k + (n-2m-4)(2m+2)] \int_0^R r^{n-2m-5} (f_k(r))^2 dr \\ &\geq \quad (\frac{n+2m}{2})^2 \int_0^R r^{n-2m-3} (f_k')^2 dr + \beta(W;R) \int_0^R r^{n-2m-1} W(x) (f_k')^2 dr \\ &+c_k[c_k + 2(\frac{n-\lambda-4}{2})^2 + (n-2m-4)(2m+2)] \int_0^R r^{n-2m-5} (f_k(r))^2 dr, \end{aligned}$$

where we have used the fact that $c_k \geq 0$ to get the above inequality. We have

$$\begin{aligned}
\frac{1}{n\omega_n}\int_{R^n}\frac{|\Delta u_k|^2}{|x|^{2m}}dx &\geq \beta_{n,m}\int_0^R r^{n-2m-5}(f_k)^2dr \\
&+\beta(W;R)\frac{(n+2m)^2}{4}\int_0^R r^{n-2m-3}W(x)(f_k)^2dr \\
&+\beta(W;R)\int_0^R r^{n-2m-1}W(x)(f_k')^2dr \\
&\geq \beta_{n,m}\int_0^R r^{n-2m-5}(f_k)^2dr \\
&+\beta(W;R)(\frac{(n+2m)^2}{4}+\frac{(n-2m-\lambda-2)^2}{4})\int_0^R r^{n-2m-3}W(x)(f_k)^2dr \\
&\geq \frac{\beta_{n,m}}{n\omega_n}\int_B \frac{u_k^2}{|x|^{2m+4}}dx \\
&+\frac{\beta(W;R)}{n\omega_n}(\frac{(n+2m)^2}{4}+\frac{(n-2m-\lambda-2)^2}{4})\int_B \frac{W(x)}{|x|^{2m+2}}u_k^2dx,
\end{aligned}$$

by Theorem 2.9. Hence, (94) holds and the proof is complete. □

Theorem 3.15 *Assume $-1 < m \leq \frac{n-4}{2}$ and let $W(x)$ be a Bessel potential on a ball B of radius R and centered at zero in R^n $(n \geq 1)$. Then there exists $C > 0$ such that the following holds for all $u \in C_0^\infty(B)$:*

$$\int_B \frac{|\Delta u|^2}{|x|^{2m}}dx \geq \frac{(n+2m)^2(n-2m-4)^2}{16}\int_B \frac{u^2}{|x|^{2m+4}}dx \tag{95}$$

$$+\beta(W;R)\frac{(n+2m)^2}{4}\int_B \frac{W(x)}{|x|^{2m+2}}u^2dx+\beta(|x|^{2m};R)||u||_{H_0^1}. \tag{96}$$

Proof: Decomposing again $u \in C_0^\infty(B_R)$ into its spherical harmonics $\Sigma_{k=0}^\infty u_k$ where $u_k = f_k(|x|)\varphi_k(x)$, we calculate

$$\begin{aligned}
\frac{1}{n\omega_n}\int_{R^n}\frac{|\Delta u_k|^2}{|x|^{2m}}dx &= \int_0^R r^{n-2m-1}(f_k''(r))^2dr+[(n-1)(2m+1)+2c_k]\int_0^R r^{n-2m-3}(f_k')^2dr \\
&+ c_k[c_k+(n-2m-4)(2m+2)]\int_0^R r^{n-2m-5}(f_k(r))^2dr \\
&\geq (\frac{n+2m}{2})^2\int_0^R r^{n-2m-3}(f_k')^2dr+\beta(|x|^{2m};R)\int_0^R r^{n-1}(f_k')^2dr \\
&+ c_k\int_0^R r^{n-2m-3}(f_k')^2dr \\
&\geq \frac{(n+2m)^2(n-2m-4)^2}{16}\int_0^R r^{n-2m-5}(f_k)^2dr \\
&+\beta(W;R)\frac{(n+2m)^2}{4}\int_0^R W(r)r^{n-2m-3}(f_k)^2dr \\
&+ \beta(|x|^{2m};R)\int_0^R r^{n-1}(f_k')^2dr+c_k\beta(|x|^{2m};R)\int_0^R r^{n-3}(f_k)^2dr \\
&= \frac{(n+2m)^2(n-2m-4)^2}{16n\omega_n}\int_{R^n}\frac{u_k^2}{|x|^{2m+4}}dx \\
&+ \frac{\beta(W;R)}{n\omega_n}(\frac{(n+2m)^2}{4})\int_{R^n}\frac{W(x)}{|x|^{2m+2}}u_k^2dx+\beta(|x|^{2m};R)||u_k||_{W_0^{1,2}}.
\end{aligned}$$

Hence (95) holds. □

We note that even for $m = 0$ and $n \geq 4$, Theorem 3.15 improves considerably Theorem A.2. in [2].

4 Higher order Rellich inequalities

In this section we will repeat the results obtained in the previous section to derive higher order Rellich inequalities with corresponding improvements. Let W be a Bessel potential, $\beta_{n,m}$ be defined as in Theorem 3.14 and

$$\sigma_{n,m} = \beta(W;R)(\frac{(n+2m)^2}{4} + \frac{(n-2m-\lambda-2)^2}{4}).$$

For the sake of convenience we make the following convention: $\prod_{i=1}^{0} a_i = 1$.

Theorem 4.1 *Let B_R be a ball of radius R and W be a Bessel potential on B_R such that $\frac{W(r)}{W_r(r)} = -\frac{\lambda}{r} + f(r)$, where $f(r) \geq 0$ and $\lim_{r\to 0} rf(r) = 0$. Assume $m \in N$, $1 \leq l \leq m$, and $2k + 4m \leq n$. Then the following inequality holds for all $u \in C_0^\infty(B_R)$*

$$\int_{B_R} \frac{|\Delta^m u|^2}{|x|^{2k}} dx \geq \prod_{i=0}^{l-1} \beta_{n,k+2i} \int_{B_R} \frac{|\Delta^{m-l} u|^2}{|x|^{2k+4l}} dx + \sum_{i=0}^{l-1} \sigma_{n,k+2i} \prod_{j=1}^{l-1} \beta_{n,k+2j-2} \int_{B_R} \frac{W(x)|\Delta^{m-i-1}u|^2}{|x|^{2k+4i+2}} dx \quad (97)$$

Proof: Follows directly from theorem 3.14. □

Theorem 4.2 *Let B_R be a ball of radius R and W be a Bessel potential on B_R such that $\frac{W(r)}{W_r(r)} = -\frac{\lambda}{r} + f(r)$, where $f(r) \geq 0$ and $\lim_{r\to 0} rf(r) = 0$. Assume $m \in N$, $1 \leq l \leq m$, and $2k+4m+2 \leq n$. Then the following inequality holds for all $u \in C_0^\infty(B_R)$*

$$\begin{aligned} \int_{B_R} \frac{|\nabla \Delta^m u|^2}{|x|^{2k}} dx &\geq (\frac{n-2k-2}{2})^2 \prod_{i=0}^{l-1} \beta_{n,k+2i+1} \int_{B_R} \frac{|\Delta^{m-l}u|^2}{|x|^{2k+4l+2}} dx \\ &+ (\frac{n-2k-2}{2})^2 \sum_{i=0}^{l-1} \sigma_{n,k+2i+1} \prod_{j=1}^{l-1} \beta_{n,k+2j-1} \int_{B_R} \frac{W(x)|\Delta^{m-i-1}u|^2}{|x|^{2k+4i+4}} dx \\ &+ \beta(W;R) \int_{B_R} W(x) \frac{|\Delta^m u|^2}{|x|^{2k}} dx \end{aligned} \quad (98)$$

Proof: Follows directly from Theorem 2.15 and the previous theorem. □

Remark 4.3 *For $k = 0$ Theorems 4.1 and 4.2 include Theorem 1.9 in [24] as a special case.*

Theorem 4.4 *Let B_R be a ball of radius R and W be a Bessel potential on B_R such that $\frac{W(r)}{W_r(r)} = -\frac{\lambda}{r} + f(r)$, where $f(r) \geq 0$ and $\lim_{r\to 0} rf(r) = 0$. Assume $m \in N$, $1 \leq l \leq m-1$, and $2k+4m \leq n$. Then the following inequality holds for all $u \in C_0^\infty(B_R)$*

$$\begin{aligned} \int_{B_R} \frac{|\Delta^m u|^2}{|x|^{2k}} dx &\geq a_{n,k}(\frac{n-2k-4}{2})^2 \prod_{i=0}^{l-1} \beta_{n,k+2i+2} \int_{B_R} \frac{|\Delta^{m-l-1}u|^2}{|x|^{2k+4l+4}} dx \\ &+ a_{n,k}(\frac{n-2k-4}{2})^2 \sum_{i=0}^{l-1} \sigma_{n,k+2i+2} \prod_{j=1}^{l-1} \beta_{n,k+2j} \int_{B_R} \frac{W(x)|\Delta^{m-i-2}u|^2}{|x|^{2k+4i+6}} dx \\ &+ \beta(W;R) a_{n,k} \int_{B_R} W(x) \frac{|\Delta^{m-1}u|^2}{|x|^{2k+2}} dx + \beta(W;R) \int_{B_R} W(x) \frac{|\nabla \Delta^{m-1}u|^2}{|x|^{2k}} dx \end{aligned} \quad (99)$$

where $a_{n,m}$ is defined in Theorem 3.10.

Proof: Follows directly from Theorem 3.10 and the previous theorem. □

The following improves Theorem 1.10 in [24] in many ways, since it is assumed there that $l \leq \frac{-n+8+2\sqrt{n^2-n+1}}{12}$ and $4m < n$. Even for $k = 0$, Theorem 4.5 below shows that we can drop the first condition and replace the second one by $4m \leq n$.

Theorem 4.5 *Let B_R be a ball of radius R and W be a Bessel potential on B_R such that . Assume $m \in N$, $1 \leq l \leq m$, and $2k+4m \leq n$. Then the following inequality holds for all $u \in C_0^\infty(B_R)$*

$$\begin{aligned}\int_{B_R} \frac{|\Delta^m u|^2}{|x|^{2k}}dx &\geq \prod_{i=1}^{l} \frac{a_{n,k+2i-2}(n-2k-4i)^2}{4} \int_{B_R} \frac{|\Delta^{m-l}u|^2}{|x|^{2k+4l}}dx \\ &+ \beta(W;R)\sum_{i=1}^{l}\prod_{j=1}^{l-1} \frac{a_{n,k+2j-2}(n-2k-4j)^2}{4} \int_{B_R} W(x)\frac{|\nabla\Delta^{m-i}u|^2}{|x|^{2k+4i-4}}dx \\ &+ \beta(W;R)\sum_{i=1}^{l} a_{n,k+2i-2}\prod_{j=1}^{l-1} \frac{a_{n,k+2j-2}(n-2k-4j)^2}{4} \int_{B_R} W(x)\frac{|\Delta^{m-i}u|^2}{|x|^{2k+4i-2}}dx,\end{aligned} \tag{100}$$

where $a_{n,m}$ are the best constants in inequality (85).

Proof: Follows directly from Theorem 3.10. □

5 Appendix (A): The class of Bessel potentials

The Bessel equation associated to a potential W

$$(B_W) \qquad y'' + \tfrac{1}{r}y' + W(r)y = 0$$

is central to all results revolving around the inequalities of Hardy and Hardy-Rellich type. We summarize in this appendix the various properties of these equations that were used throughout this paper.

Definition 1 *We say that a non-negative real valued C^1-function is a Bessel potential on $(0,R)$ if there exists $c>0$ such that the equation (B_{cW}) has a positive solution on $(0,R)$.*
The class of Bessel potentials on $(0,R)$ will be denoted by $\mathcal{B}(0,R)$.

Note that the change of variable $z(s) = y(e^{-s})$ maps the equation $y'' + \frac{1}{r}y' + W(r)y = 0$ into

$$(B'_W) \qquad z'' + e^{-2s}W(e^{-s})z(s) = 0. \tag{101}$$

On the other hand, the change of variables $\psi(t) = \frac{-e^{-t}y'(e^{-t})}{y(e^{-t})}$ maps it into the nonlinear equation

$$(B''_W) \qquad \psi'(t) + \psi^2(t) + e^{-2t}W(e^{-t}) = 0. \tag{102}$$

This will allow us to relate the existence of positive solutions of (B_W) to the non-oscillatory behaviour of equations (B'_W) and (B''_W).
The theory of sub/supersolutions –applied to (B''_W) (See Wintner [28, 29, 16])– already yields, that if (B_W) has a positive solution on an interval $(0,R)$ for some non-negative potential $W \geq 0$, then for any W such that $0 \leq V \leq W$, the equation (B_V) has also a positive solution on $(0,R)$. This leads to the definition of the *weight* of a potential $W \in \mathcal{B}(0,R)$ as:

$$\beta(W;R) = \sup\{c>0;\ (B_{cW}) \text{ has a positive solution on } (0,R)\}. \tag{103}$$

The following is now straightforward.

Proposition 5.1 *1) The class $\mathcal{B}(0,R)$ is a closed convex and solid subset of $C^1(0,R)$.*
2) For every $W \in \mathcal{B}(0,R)$, the equation

$$(B_W) \qquad y'' + \tfrac{1}{r}y' + \beta(W;R)W(r)y = 0$$

has a positive solution on $(0,R)$.

The following gives an integral criteria for Bessel potentials.

Proposition 5.2 *Let W be a positive locally integrable function on $\mathbb{R}$.*

1. *If $\liminf_{r\to 0} \ln(r)\int_0^r sW(s)ds > -\infty$, then for every $R>0$, there exists $\alpha := \alpha(R) > 0$ such that the scaled function $W_\alpha(x) := \alpha^2 W(\alpha x)$ is a Bessel potential on $(0,R)$.*

2. *If $\lim_{r\to 0} \ln(r)\int_0^r sW(s)ds = -\infty$, then there are no $\alpha, c > 0$, for which $W_{\alpha,c} = cW(\alpha|x|)$ is a Bessel potential on $(0,R)$.*

Proof: This relies on well known results concerning the existence of non-oscillatory solutions (i.e., those $z(s)$ such that $z(s) > 0$ for $s > 0$ sufficiently large) for the second order linear differential equations

$$z''(s) + a(s)z(s) = 0, \tag{104}$$

where a is a locally integrable function on $\mathbb{R}$. For these equations, the following integral criteria are available. We refer to [16, 17, 28, 29, 30]) among others for proofs and related results.

i) If $\limsup_{t\to\infty} t\int_t^\infty a(s)ds < \frac{1}{4}$, then Eq. (104) is non-oscillatory.

ii) If $\liminf_{t\to\infty} t\int_t^\infty a(s)ds > \frac{1}{4}$, then Eq. (104) is oscillatory.

It follows that if $\liminf_{r\to 0} \ln(r)\int_0^r sW(s)ds > -\infty$ holds, then there exists $\delta > 0$ such that (B_W) has a positive solution on $(0,\delta)$. An easy scaling argument then shows that there exists $\alpha > 0$ such that $W_\alpha(x) := \alpha^2 W(\alpha x)$ is a Bessel potential on $(0,R)$. The rest of the proof is similar. $\square$

We now exhibit a few explicit Bessel potentials and compute their weights. We use the following notation.

$$log^{(1)}(.) = log(.) \quad \text{and} \quad log^{(k)}(.) = log(log^{(k-1)}(.)) \quad \text{for} \quad k \geq 2. \tag{105}$$

and

$$X_1(t) = (1 - \log(t))^{-1}, \quad X_k(t) = X_1(X_{k-1}(t)) \quad k = 2, 3, ..., \tag{106}$$

Theorem 5.1 Explicit Bessel potentials

1. *$W \equiv 0$ is a Bessel potential on $(0,R)$ for any $R>0$.*

2. *The Bessel function J_0 is a positive solution for equation (B_W) with $W \equiv 1$, on $(0, z_0)$, where $z_0 = 2.4048...$ is the first zero of J_0. Moreover, z_0 is larger than the first root of any other solution for (B_1). In other words,*

   for every $R>0$,
$$\beta(1;R) = \frac{z_0^2}{R^2}. \tag{107}$$

3. *If $a<2$, then there exists $R_a > 0$ such that $W(r) = r^{-a}$ is a Bessel potential on $(0, R_a)$.*

4. *For each $k \geq 1$ and $\rho > R(e^{e^{e^{\cdot^{\cdot^{e(k-times)}}}}})$, the equation $(B_{\frac{1}{4}W_{k,\rho}})$ corresponding to the potential*
$$W_{k,\rho}(r) = \Sigma_{j=1}^k U_j \text{ where } U_j(r) = \tfrac{1}{r^2}\big(\textstyle\prod_{i=1}^j log^{(i)}\frac{\rho}{r}\big)^{-2}$$
   has a positive solution on $(0,R)$ that is explicitly given by $\varphi_{k,\rho}(r) = (\prod_{i=1}^k log^{(i)}\frac{\rho}{r})^{\frac{1}{2}}$. On the other hand, the equation $(B_{\frac{1}{4}W_{k,\rho}+\lambda U_k})$ corresponding to the potential $\frac{1}{4}W_{k,\rho} + \lambda U_k$ has no positive solution for any $\lambda > 0$. In other words, $W_{k,\rho}$ is a Bessel potential on $(0,R)$ with
$$\beta(W_{k;\rho}, R) = \tfrac{1}{4} \text{ for any } k \geq 1. \tag{108}$$

5. *For each $k \geq 1$ and $R > 0$, the equation $(B_{\frac{1}{4}\tilde{W}_{k,R}})$ corresponding to the potential*

$$\tilde{W}_{k,R}(r) = \Sigma_{j=1}^{k}\tilde{U}_j \text{ where } \tilde{U}_j(r) = \tfrac{1}{r^2}X_1^2(\tfrac{r}{R})X_2^2(\tfrac{r}{R})\dots X_{j-1}^2(\tfrac{r}{R})X_j^2(\tfrac{r}{R})$$

has a positive solution on $(0, R)$ that is explicitly given by

$$\varphi_k(r) = (X_1(\frac{r}{R})X_2(\frac{r}{R})\dots X_{k-1}(\frac{r}{R})X_k(\frac{r}{R}))^{-\frac{1}{2}}.$$

On the other hand, the equation $(B_{\frac{1}{4}\tilde{W}_{k,R}+\lambda\tilde{U}_k})$ corresponding to the potential $\frac{1}{4}\tilde{W}_{k,R} + \lambda\tilde{U}_k$ has no positive solution for any $\lambda > 0$. In other words, $\tilde{W}_{k,R}$ is a Bessel potential on $(0, R)$ with

$$\beta(\tilde{W}_{k,R}; R) = \tfrac{1}{4} \quad \text{for any } k \geq 1. \tag{109}$$

Proof: 1) It is clear that $\varphi(r) = -log(\frac{e}{R}r)$ is a positive solution of (B_0) on $(0, R)$ for any $R > 0$.
2)The best constant for which the equation $y'' + \frac{1}{r}y' + cy = 0$ has a positive solution on $(0, R)$ is $\frac{z_0^2}{R^2}$, where $z_0 = 2.4048...$ is the first zero of Bessel function $J_0(z)$. Indeed if α is the first root of the an arbitrary solution of the Bessel equation $y'' + \frac{y'}{r} + y(r) = 0$, then we have $\alpha \leq z_0$. To see this let $x(t) = aJ_0(t) + bY_0(t)$, where J_0 and Y_0 are the two standard linearly independent solutions of Bessel equation, and a and b are constants. Assume the first zero of $x(t)$ is larger than z_0. Since the first zero of Y_0 is smaller than z_0, we have $a \geq 0$. Also $b \leq 0$, because $Y_0(t) \to -\infty$ as $t \to 0$. Finally note that $Y_0(z_0) > 0$, so if $b < 0$, then $x(z_0 + \epsilon) < 0$ for ϵ sufficiently small. Therefore, $b = 0$ which is a contradiction.
3) follows directly from the integral criteria.
4) That φ_k is an explicit solution of the equation $(B_{\frac{1}{4}W_k})$ is straightforward. Assume now that there exists a positive function φ such that

$$-\frac{\varphi'(r) + r\varphi''(r)}{\varphi(r)} = \frac{1}{4}\sum_{j=1}^{k-1}\frac{1}{r}(\prod_{i=1}^{j} log^{(i)}\frac{\rho}{r})^{-2} + (\frac{1}{4} + \lambda)\frac{1}{r}(\prod_{i=1}^{k} log^{(i)}\frac{\rho}{r})^{-2}.$$

Define $f(r) = \frac{\varphi(r)}{\varphi_k(r)} > 0$, and calculate,

$$\frac{\varphi'(r) + r\varphi''(r)}{\varphi(r)} = \frac{\varphi_k'(r) + r\varphi_k''(r)}{\varphi_k(r)} + \frac{f'(r) + rf''(r)}{f(r)} - \frac{f'(r)}{f(r)}\sum_{i=1}^{k}\frac{1}{\prod_{j=1}^{i}\log^j(\frac{\rho}{r})}.$$

Thus,

$$\frac{f'(r) + rf''(r)}{f(r)} - \frac{f'(r)}{f(r)}\sum_{i=1}^{k}\frac{1}{\prod_{j=1}^{i}\log^j(\frac{\rho}{r})} = -\lambda\frac{1}{r}(\prod_{i=1}^{k} log^{(i)}\frac{\rho}{r})^{-2}. \tag{110}$$

If now $f'(\alpha_n) = 0$ for some sequence $\{\alpha_n\}_{n=1}^{\infty}$ that converges to zero, then there exists a sequence $\{\beta_n\}_{n=1}^{\infty}$ that also converges to zero, such that $f''(\beta_n) = 0$, and $f'(\beta_n) > 0$. But this contradicts (110), which means that f is eventually monotone for r small enough. We consider the two cases according to whether f is increasing or decreasing:

Case I: Assume $f'(r) > 0$ for $r > 0$ sufficiently small. Then we will have

$$\frac{(rf'(r))'}{rf'(r)} \leq \sum_{i=1}^{k}\frac{1}{r\prod_{j=1}^{i}\log^j(\frac{\rho}{r})}.$$

Integrating once we get

$$f'(r) \geq \frac{c}{r\prod_{j=1}^{k}\log^j(\frac{\rho}{r})},$$

for some $c > 0$. Hence, $\lim_{r\to 0} f(r) = -\infty$ which is a contradiction.

Case II: Assume $f'(r) < 0$ for $r > 0$ sufficiently small. Then

$$\frac{(rf'(r))'}{rf'(r)} \geq \sum_{i=1}^{k} \frac{1}{r\prod_{j=1}^{i} \log^{j}(\frac{\rho}{r})}.$$

Thus,

$$f'(r) \geq -\frac{c}{r\prod_{j=1}^{k} \log^{j}(\frac{\rho}{r})}, \tag{111}$$

for some $c > 0$ and $r > 0$ sufficiently small. On the other hand

$$\frac{f'(r) + rf''(r)}{f(r)} \leq -\lambda \sum_{j=1}^{k} \frac{1}{r}(\prod_{i=1}^{j} log^{(i)} \frac{R}{r})^{-2} \leq -\lambda(\frac{1}{\prod_{j=1}^{k} \log^{j}(\frac{\rho}{r})})'.$$

Since $f'(r) < 0$, there exists l such that $f(r) > l > 0$ for $r > 0$ sufficiently small. From the above inequality we then have

$$bf'(b) - af'(a) < -\lambda l(\frac{1}{\prod_{j=1}^{k} \log^{j}(\frac{\rho}{b})} - \frac{1}{\prod_{j=1}^{k} \log^{j}(\frac{\rho}{a})}).$$

From (111) we have $\lim_{a\to 0} af'(a) = 0$. Hence,

$$bf'(b) < -\frac{\lambda l}{\prod_{j=1}^{k} \log^{j}(\frac{\rho}{b})},$$

for every $b > 0$, and

$$f'(r) < -\frac{\lambda l}{r\prod_{j=1}^{k} \log^{j}(\frac{\rho}{r})},$$

for $r > 0$ sufficiently small. Therefore,

$$\lim_{r\to 0} f(r) = +\infty,$$

and by choosing l large enouph (e.g., $l > \frac{c}{\lambda}$) we get to contradict (111).
The proof of 5) is similar and is left to the interested reader. $\square$

6 Appendix (B): The evaluation of $a_{n,m}$

Here we evaluate the best constants $a_{n,m}$ which appear in Theorem 3.10.

Theorem 6.1 *Suppose $n \geq 1$ and $m \leq \frac{n-2}{2}$. Then for any $R > 0$, the constants*

$$a_{n,m} = \inf\left\{ \frac{\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} dx}{\int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx}; u \in C_0^\infty(B_R) \setminus \{0\} \right\}$$

are given by the following expressions.

1. *For $n = 1$*
   - *if $m \in (-\infty, -\frac{3}{2}) \cup [-\frac{7}{6}, -\frac{1}{2}]$, then*
   $$a_{1,m} = (\frac{1+2m}{2})^2$$
   - *if $-\frac{3}{2} < m < -\frac{7}{6}$, then*
   $$a_{1,m} = \min\{(\frac{n+2m}{2})^2, \frac{(\frac{(n-4-2m)(n+2m)}{4} + 2)^2}{(\frac{n-4-2m}{2})^2 + 2}\}.$$

2. *If* $m = \frac{n-4}{2}$, *then*
$$a_{m,n} = \min\{(n-2)^2, n-1\}.$$

3. *If* $n \geq 2$ *and* $m \leq \frac{-(n+4)+2\sqrt{n^2-n+1}}{6}$, *then* $a_{n,m} = (\frac{n+2m}{2})^2$.

4. *If* $2 \leq n \leq 3$ *and* $\frac{-(n+4)+2\sqrt{n^2-n+1}}{6} < m \leq \frac{n-2}{2}$, *or* $n \geq 4$ *and* $\frac{n-4}{2} < m \leq \frac{n-2}{2}$, *then*
$$a_{n,m} = \frac{(\frac{(n-4-2m)(n+2m)}{4} + n - 1)^2}{(\frac{n-4-2m}{2})^2 + n - 1}.$$

5. *For* $n \geq 4$ *and* $\frac{-(n+4)+2\sqrt{n^2-n+1}}{6} < m < \frac{n-4}{2}$, *define* $k^* = [(\frac{\sqrt{3}}{3} - \frac{1}{2})(n-2)]$.

   - *If* $k^* \leq 1$, *then*
$$a_{n,m} = \frac{(\frac{(n-4-2m)(n+2m)}{4} + n - 1)^2}{(\frac{n-4-2m}{2})^2 + n - 1}.$$

   - *For* $k^* > 1$ *the interval* $(m_0^1 := \frac{-(n+4)+2\sqrt{n^2-n+1}}{6}, m_0^2 := \frac{n-4}{2})$ *can be divided in* $2k^* - 1$ *subintervals. For* $1 \leq k \leq k^*$ *define*
$$m_k^1 := \frac{2(n-5) - \sqrt{(n-2)^2 - 12k(k+n-2)}}{6},$$
$$m_k^2 := \frac{2(n-5) + \sqrt{(n-2)^2 - 12k(k+n-2)}}{6}.$$
   If $m \in (m_0^1, m_1^1] \cup [m_1^2, m_0^2)]$, *then*
$$a_{n,m} = \frac{(\frac{(n-4-2m)(n+2m)}{4} + n - 1)^2}{(\frac{n-4-2m}{2})^2 + n - 1}.$$

   - *For* $k \geq 1$ *and* $m \in (m_k^1, m_{k+1}^1] \cup [m_{k+1}^2, m_k^2)$, *then*
$$a_{n,m} = \min\{\frac{(\frac{(n-4-2m)(n+2m)}{4} + k(n+k-2))^2}{(\frac{n-4-2m}{2})^2 + k(n+k-2)}, \frac{(\frac{(n-4-2m)(n+2m)}{4} + (k+1)(n+k-1))^2}{(\frac{n-4-2m}{2})^2 + (k+1)(n+k-1)}\}.$$
   For $m \in (m_{k^*}^1, m_{k^*}^2)$, *then*
$$a_{n,m} = \min\{\frac{(\frac{(n-4-2m)(n+2m)}{4} + k^*(n+k^*-2))^2}{(\frac{n-4-2m}{2})^2 + k^*(n+k^*-2)}, \frac{(\frac{(n-4-2m)(n+2m)}{4} + (k^*+1)(n+k^*-1))^2}{(\frac{n-4-2m}{2})^2 + (k^*+1)(n+k^*-1)}\}.$$

Proof: Letting $V(r) = r^{-2m}$ then,
$$W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r) = ((\frac{n-2m-2}{2})^2 - 2 - 4m - 2m(2m+1))r^{-2m-2}.$$
In order to satisfy condition (69) we should have
$$\frac{-(n+4)+2\sqrt{n^2-n+1}}{6} \leq m \leq \frac{-(n+4)+2\sqrt{n^2-n+1}}{6}. \tag{112}$$
So, by Theorem 3.3 under the above condition we have $a_{n,m} = (\frac{n+2m}{2})^2$ as in the radial case.
For the rest of the proof we will use an argument similar to that of Theorem 6.4 in [24] who computed $a_{n,m}$ in the case where $n \geq 5$ and for certain intervals of m.
Decomposing again $u \in C_0^\infty(B_R)$ into spherical harmonics; $u = \Sigma_{k=0}^\infty u_k$, where $u_k = f_k(|x|)\varphi_k(x)$, one has
$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|\Delta u_k|^2}{|x|^{2m}} dx &= \int_{\mathbb{R}^n} |x|^{-2m}(f_k''(|x|))^2 dx + ((n-1)(2m+1) + 2c_k) \int_{\mathbb{R}^n} |x|^{-2m-2}(f_k')^2 dx \\ &+ c_k(c_k + (n-4-2m)(2m+2)) \int_{\mathbb{R}^n} |x|^{-2m-4}(f_k)^2 dx,\end{aligned} \tag{113}$$

$$\int_{\mathbb{R}^n} \frac{|\nabla u_k|^2}{|x|^{2m+2}} dx = \int_{\mathbb{R}^n} |x|^{-2m-2} (f_k')^2 dx + c_k \int_{\mathbb{R}^n} |x|^{-2m-4} (f_k)^2 dx. \tag{114}$$

One can then prove as in [24] that

$$a_{n,m} = \min \{A(k,m,n);\ k \in \mathbb{N}\} \tag{115}$$

where

$$A(k,m,n) = \frac{(\frac{(n-4-2m)(n+2m)}{4} + c_k)^2}{(\frac{n-4-2m}{2})^2 + c_k} \text{ if } m = \frac{n-4}{2} \tag{116}$$

and

$$A(k,m,n) := c_k \text{ if } m = \frac{n-4}{2} \text{ and } n+k > 2. \tag{117}$$

Note that when $m = \frac{n-4}{2}$ and $n+k > 2$, then $c_k \neq 0$. Actually, this also holds for $n+k \leq 2$, in which case one deduces that if $m = \frac{n-4}{2}$, then

$$a_{n,m} = \min\{(n-2)^2 = (\frac{n+2m}{2})^2, (n-1) = c_1\}$$

which is statement 2).

The rest of the proof consists of computing the infimum especially in the cases not considered in [24]. For that we consider the function

$$f(x) = \frac{(\frac{(n-4-2m)(n+2m)}{4} + x)^2}{(\frac{n-4-2m}{2})^2 + x}.$$

It is easy to check that $f'(x) = 0$ at x_1 and x_2, where

$$x_1 = -\frac{(n-4-2m)(n+2m)}{4} \tag{118}$$

$$x_2 = \frac{(n-4-2m)(-n+6m+8)}{4}. \tag{119}$$

Observe that for for $n \geq 2$, $\frac{n-8}{6} \leq \frac{n-4}{2}$. Hence, for $m \leq \frac{n-8}{6}$ both x_1 and x_2 are negative and hence $a_{n,m} = (\frac{n+2m}{2})^2$. Also note that

$$\frac{-(n+4) - 2\sqrt{n^2-n+1}}{6} \leq \frac{n-8}{6} \quad for \quad all \quad n \geq 1.$$

Hence, under the condition in 3) we have $a_{n,m} = (\frac{n+2m}{2})^2$.

Also for $n = 1$ if $m \leq -\frac{3}{2}$ both critical points are negative and we have $a_{1,m} \leq (\frac{1+2m}{2})^2$. Comparing $A(0,m,n)$ and $A(1,m,n)$ we see that $A(1,m,n) \geq A(0,m,n)$ if and only if (112) holds.

For $n = 1$ and $-\frac{3}{2} < m < -\frac{7}{6}$ both x_1 and x_2 are positive. Consider the equations

$$x(x-1) = x_1 = \frac{(2m+3)(2m+1)}{4},$$

and

$$x(x-1) = x_2 = -\frac{(2m+3)(6m+7)}{4}.$$

By simple calculations we can see that all four solutions of the above two equations are less that two. Since, $A(1,m,1) < A(0,m,1)$ for $m < -\frac{7}{6}$, we have $a_{1,m} \leq \min\{A(1,m,1), A(2,m,1)\}$ and 1) follows.

For $n \geq 2$ and $\frac{n-4}{2} < m < \frac{n-2}{2}$ we have $x_1 > 0$ and $x_2 < 0$. Consider the equation

$$x(x+n-2) = x_1 = -\frac{(n-4-2m)(n+2m)}{4}.$$

Then $\frac{2m+4-n}{2}$ and $-\frac{(2m+n)}{2}$ are solutions of the above equation and both are less than one. Since, for $n \geq 4$

$$\frac{n-2}{2} > \frac{-(n+4) + 2\sqrt{n^2-n+1}}{6},$$

and $A(1,m,n) \leq A(0,m,n)$ for $m \geq \frac{-(n+4)+2\sqrt{n^2-n+1}}{6}$, the best constant is equal to what 4) claims.

5) follows from an argument similar to that of Theorem 6.4 in [24].

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